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Gravitational Waves and Black Holes: Beyond the Mirror

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ABSTRACT

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Index Terms: adjoint sequence • differential sequence • gravitational waves • kerr metric • killing operator • lie algebroid • minkowski metric • riemann operator • schwarzschild metric • spencer operator

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Keywords: *adjoint sequence, differential sequence, gravitational waves, kerr metric, killing operator, lie algebroid, minkowski metric, riemann operator, schwarzschild metric, spencer operator*

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ï»¿

1 INTRODUCTION

When M. Janet introduced in 1920 the first finite length differential sequence as a footnote of his paper [1], he surely did not know about the possibility to use such a sequence in elasticity theory along the way introduced by the brothers E. and F. Cosserat in 1909 [2]. Being a visiting student of D. C. Spencer (1912-2001) at Princeton University in 1970, I discovered that he was not even knowing the mathematical foundations of general relativity (GR) studied by his close friend J. A. Wheeler (1911-2008) who was offering 1000 dollars at that time to anybody finding a potential for Einstein equations in vacuum. This possibility is known to exist for Maxwell equations in electromagnetism (EM), usually defined by $dA = F$ while introducing the exterior derivative. I discovered in 1995 the negative solution of this challenge by using (formal) adjoint operators in a systematic way (See ideXlab on the net !), contrary to the general belief of the GR community. As a byproduct, such a result can only be found in books of control theory [3, 4]. In 1980, I met Janet who was still alive and he told me about the work of E. Vessiot and the resulting “affair” concerning the Differential Galois Theory [5, 6].

In 2015, a few meetings have been organized by the “ Institut Henri Poincaré ” (IHP) in Paris during three months about “ Mathematical General relativity ” . In particular, a Celebration of the 100th Anniversary was held on 16-20 November, largely dedicated to gravitational waves (GW) and followed by two days in honor of A. Lichnerowicz on 19 + 20 December. As Lichnerowicz had been my main advisor during more than 20 years and died in 1998, I decided to participate to all these events. The atmosphere was very unpleasant because everybody knew that most sponsors should stop funding. One invited talk ” Are Black Holes Real ” given by S. Klainermann was so badly accepted that I could only answer to my neighbour, a young foreign student, that I was listening to it for the first time. The idea was to distinguish between three kinds of “Reality”, namely “Virtual reality, Physical reality and Mathematical reality”, the latter allowing to write a paper without any mathematical mistake but no comment was done on the mathematical assumptions used at the beginning [7, 8]. Less than 6 months later, LIGO announced to have detected GW produced by a couple of merging binary black holes (!) and this event, highly spread in newspapers, has been followed by the diffusion of pictures of black holes [9]. Since that time, I started to have doubts and, being specialist of control theory, I decided to use my knowledge for studying the origin of GW. In 2017, I discovered why GW cannot exist because Einstein, copying Beltrami, both ignoring that the Einstein operator, linearization of the Einstein tensor over the M metric, was surprisingly self-adjoint[10, 11]. I started to have doubts, not about the proper detection but mainly about the defining equations. Then, I started to have serious doubts when LIGO did stop for 3 years and I don't speak about the lack of any result for KAGRA after spending 250 millions of dollars. It is at this moment that I decided to care about black holes while taking into account a few recent papers I wrote about the comparison of the M,S and K metrics [12, 13] but also as a way to disagree with the approach used by L. Andersson and collaborators met while lecturing at the Albert Einstein Institute (AEI) of Potsdam (October 23-27, 2017) [14].

In the Special Relativity paper of Einstein (1905), only a footnote provides a reference to the conformal group of space-time, namely the group of transformations preserving the Minkowski metric ω up to a function factor, but there is no proof that the conformal factor should be equal to 1. Over a manifold of dimension $n \geq 3$, this group has n translations, $n(n-1)/2$ rotations, 1 dilatation and n non-linear elations introduced by E . Cartan in 1922, with a total number of $N = (n+1)(n+2)/2$ parameters that is $N = 15$ when $n = 4$ [15, 16]. However, it is also the number of the Cauchy stress equations (1823), the Cosserat couple-stress equations (1909), the only Clausius virial equation (1870), the Maxwell (1873) and Weyl (1918) equations which are among the most famous partial differential equations that can be found today in any textbook dealing with elasticity theory, continuum mechanics, thermodynamics or electromagnetism. The purpose of this paper is to prove that the form of these equations only depends on the structure of the conformal group for an arbitrary $n \geq 1$ because they are described as a whole by the (formal) adjoint of the first Spencer operator existing in the Spencer differential sequence. Such a group theoretical implication is obtained by applying totally new differential geometric methods in field theory. In particular, when $n = 4$, the main idea is to enlarge the group from 10 up to 11 or 15 parameters by using the Weyl or conformal group instead of the Poincaré group of space-time.

Contrary to the Einstein equations, these equations can be all parametrized by the adjoint of the second Spencer operator through $N = n(n-1)/2$ potentials. These results bring the need to revisit the mathematical foundations of both General Relativity (GR) and Gauge Theory (GT) according to a clever but rarely quoted paper of H. Poincaré (1901) [17]. They strengthen the comments we already made about the dual confusions made by Einstein (1915) while following Beltrami (1892), both using the same operator when $n = 3$ and $n = 4$ but ignoring it is self-adjoint in the framework of differential double duality. They also question the origin and existence of black holes.

With some more preliminary details provided in [18], we have successively:

When $n = 2$ in plane elasticity with Riemann operator $\Omega \rightarrow d_{22}\Omega_{11} + d_{11}\Omega_{22} - 2d_{12}\Omega_{12}$, G.B. Airy found in 1863 the possibility to parametrize the $Cauchy = ad(Killing)$ operator $\sigma \rightarrow$

$$\begin{array}{|lcl|} \hline d_1\sigma^{11} + d_2\sigma^{12} & = & 0 \\ d_1\sigma^{21} + d_2\sigma^{22} & = & 0 \\ \hline \end{array}$$

$$\begin{array}{ccccc} & & 1 & & \\ & \nearrow & & \searrow & \\ 2 & \stackrel{D}{\longrightarrow} & 3 & \stackrel{D_1}{\longrightarrow} & 1 \\ & & & & \\ 2 & \stackrel{ad(D)}{\longleftarrow} & 3 & \stackrel{ad(D_1)}{\longleftarrow} & 1 \end{array}$$

with $\sigma^{12} = \sigma^{21}$ by a single stress function ϕ wearing his name through the $Airy = ad(Riemann)$ operator $\phi \rightarrow (d_{22}\phi, -d_{12}\phi, d_{11}\phi)$ as follows:

$$\begin{array}{ccccccccc} & & 2 & \stackrel{D}{\longrightarrow} & 3 & \stackrel{D_1}{\longrightarrow} & 1 & \longrightarrow & 0 \\ & & & & & & & & \\ 0 & \longleftarrow & 2 & \stackrel{ad(D)}{\longleftarrow} & 3 & \stackrel{ad(D_1)}{\longleftarrow} & 1 & & \end{array}$$

Multiplying the Riemann operator on the left by ϕ and "integrating by parts", we obtain:

$$\phi(d_{22}\Omega_{11}) = (d_{22}\phi)\Omega_{11} + d_2(\phi d_2\Omega_{11} - (d_2\phi)\Omega_{11})$$

As for the fact that the Cauchy operator is the adjoint of the Killing operator, this result can be found in any textbook of elasticity.

When $n = 3$ in space elasticity, E. Beltrami found in 1892 the possibility to parametrize the $Cauchy = ad(Killing)$ operator by means of six stress functions $\phi_{ij} = \phi_{ji}$ wearing his name through the self-adjoint $Beltrami = ad(Riemann)$ operator as follows:

$$\begin{array}{ccccccccc} & & 3 & \underset{1}{\stackrel{Killing}{\longrightarrow}} & 6 & \underset{2}{\stackrel{Riemann}{\longrightarrow}} & 6 & \underset{1}{\stackrel{Bianchi}{\longrightarrow}} & 3 & \rightarrow & 0 \\ 0 & \leftarrow & 3 & \underset{1}{\stackrel{Cauchy}{\longleftarrow}} & 6 & \underset{2}{\stackrel{Beltrami}{\longleftarrow}} & 6 & \underset{1}{\stackrel{Lanczos}{\longleftarrow}} & 3 & & \end{array}$$

As can be checked, the alternate sum of dimensions in each sequence, called Euler-Poincaré characteristic, does vanish indeed. More generally, studying the Lanczos problems in 2001 for a dimension $n \geq 3$, I discovered that the $Beltrami = ad(Riemann)$ operator can be parametrized by the $Lanczos = ad(Bianchi)$ operator in the geometrical and physical adjoint sequences made by operators acting on tensors, giving order of operators and number of components as follows:

The geometrical and adjoint physical long exact differential sequences of operators acting on tensors, giving order of operators and number of components as follows:

$$
\begin{array}{ccccccccccc}
 & & n & \xrightarrow[1]{Killing} & \frac{n(n+1)}{2} & \xrightarrow[2]{Riemann} & \frac{n^2(n^2-1)}{12} & \xrightarrow[1]{Bianchi} & \frac{n^2(n^2-1)(n-2)}{24} & \rightarrow & 0 \\
0 & \leftarrow & n & \xleftarrow[1]{Cauchy} & \frac{n(n+1)}{2} & \xleftarrow[2]{Beltrami} & \frac{n^2(n^2-1)}{12} & \xleftarrow[1]{Lanczos} & \frac{n^2(n^2-1)(n-2)}{24} & &
\end{array}
$$

When $n = 4$ in space-time, Einstein, probably knowing the work of Beltrami because the comparison needs no comment ([18], Proposition 4.1, p 28), made a terrible confusion in 1915 between the $Cauchy = ad(Killing)$ operator and the div operator induced from the Bianchi operator because both have been using the same Einstein operator but ignoring that such an operator is self-adjoint in the framework of differential double duality when $n \geq 3$ (See [10] for more details).

Looking at [10] in order to understand the origin of GW or at section 4, the second order linearized Ricci operator $S_2T^* \rightarrow S_2T^* : (\Omega_{ij}) \rightarrow (R_{ij})$ with 4 terms is defined by the formula:

$$2R_{ij} = \omega^{rs}(d_{rs}\Omega_{ij} + d_{ij}\Omega_{rs} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri}) = 2R_{ji}$$

Let us now introduce the linear map $C : S_2T^* \rightarrow S_2T^* : \Omega_{ij} \rightarrow \bar{\Omega}_{ij} = \Omega_{ij} - \frac{1}{2}\omega_{ij}\omega^{rs}\Omega_{rs}$ where Ω is a perturbation of the Minkowski metric ω , invertible if and only if $n \geq 3$. It is well known that the Einstein operator is defined by the same map $C : R_{ij} \rightarrow R_{ij} - \frac{1}{2}\omega_{ij}\omega^{rs}R_{rs} = E_{ij}$ not depending on any conformal factor. Comparing to [10] or to *any other textbook*, the GW are defined by the (strange!) composite operator $\mathcal{X} : \bar{\Omega} \xrightarrow{Einstein} E$ in such a way that $Einstein : \mathcal{X} \circ C$. Taking the respective adjoint operators, remembering that $Einstein = ad(Einstein)$ and that we have $ad(P \circ Q) = ad(Q) \circ ad(P)$ whenever P, Q are two operators, we obtain:

$$Einstein = ad(C) \circ ad(\mathcal{X}) = C \circ ad(\mathcal{X}) = C \circ Ricci$$
$$\Rightarrow ad(\mathcal{X}) = Ricci \Rightarrow \mathcal{X} = ad(Ricci)$$

Introducing the test functions $\lambda^{ij} = \lambda^{ji}$ and setting as usual $\Pi = \omega^{ij}d_{ij}$, we get:

$$d_{ij}(\omega^{ij}\lambda^{rs} + \omega^{rs}\lambda^{ij} - \omega^{sj}\lambda^{ri} - \omega^{ri}\lambda^{sj}) = \sigma^{rs} \Rightarrow d_r(\sigma^{rs}) = 0$$

which is exactly the generalized version of the Beltrami parametrization described in the preceding diagram with 10 instead of 20 stress functions when $n = 4$.

However, the above results are showing out the close links existing between group theory and differential sequences. For this reason, we briefly recall the historical framework leading to these new results and the reason for which we have not been able to quote many external references. This paper is also written as a smile to the famous English writer Lewis Carroll who used these words in the novel he wrote in 1871, six years after " Alice in Wonderland".

The concept of " group", introduced in mathematics for the first time by E. Galois (1830), slowly passed from algebra to geometry with the work of S. Lie on Lie groups (1880) and Lie pseudogroups (1890) of transformations. The concept of a finite length differential sequence, now called Janet sequence, has been described for the first time as a footnote by M. Janet (1920). Then, the work of D. C. Spencer (1970) has been the first attempt to use the formal theory of systems of partial differential equations in order to study the formal theory of Lie pseudogroups [20, 21].

However, the linear and nonlinear Spencer sequences for Lie pseudogroups, though never used in physics, largely supersede the " Cartan structure equations " (1905) and are quite different from the " Vessiot structure equations " (1903), introduced for the same purpose but still not known today because they have never been acknowledged by E. Cartan and successors [5, 6]. This diagram explains why it has never been possible to connect Gauge Theory (GT) using Maurer-Cartan (MC) equations with torsion $^+$ curvature with GR using (Riemann curvature alone:

$$
\begin{array}{ccccccc}
 & & CARTAN & \longrightarrow & SPENCER & & GT \\
 & \nearrow & & & & & \\
LIE & & \updownarrow & Cauchy, Cosserat, Maxwell & \updownarrow & & \\
 & \searrow & & & & & \\
 & & VESSIOT & \longrightarrow & JANET & & GR
\end{array}
$$

Example 1.1: In order to explain the difference existing between a Lie group and a Lie pseudogroup of transformations, let us consider the Lie group of projective transformations of the real line and differentiate three times the local action law as follows with $a, b, c, d = cst$ with $d \neq 0$:

$$y = \frac{ax+b}{cx+d}$$
$$\Rightarrow \quad y_x = \frac{ad-bc}{(cx+d)^2}$$
$$\Rightarrow \quad y_{xx} = -2\frac{(ad-bc)c}{(cx+d)^3}$$
$$\Rightarrow \quad y_{xxx} = 6\frac{(ad-bc)c^2}{(cx+d)^4}$$

and get the single Schwarzian third order nonlinear OD equation $(y_{xxx}/y_x) - (3/2)(y_{xx}/y_x)^2 = 0$. Setting then $y = x + t\xi + ...$ and linearizing at the identity $y = x$ when $t0$ we obtain the infinitesimal Lie equation $\xi_{xxx} = 0 \Leftrightarrow \partial_{xxx}\xi = 0$

$$\begin{array}{ccccccccccc} & & 0 & \to & \Theta & \overset{j_3}{\to} & 3 & \overset{d_1}{\to} & 3 & \to & 0 \\ & & & & & & \downarrow & & \| & & \\ & & 0 & \to & 1 & \overset{j_3}{\to} & 4 & \overset{d_1}{\to} & 3 & \to & 0 \\ & & & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \Theta & \to & 1 & \overset{D}{\to} & 1 & \longrightarrow & 0 & & \end{array}$$

with a basis of solutions $\{\theta_\tau\} = \{\partial_x, x\partial_x, \frac{1}{2}x^2\partial_x\}$ generating the space of solutions $\Theta \subset T$ over the constants with $[\Theta, \Theta] \subset \Theta$. We have the three abelian subgroups $y = ax, y = x + b, \frac{1}{y} = \frac{1}{x} + c$ $n = 1$ with no rotation but $N = 3$. However, the next examples will prove that no group theoretical technique may be applied when solutions cannot be known.

Example 1.2: With $m = n = 2, q = 1$ and $\omega = (\alpha \in T^*, \beta \in \wedge^2 T^*,)$, let us consider the Lie operator $\mathcal{D} : T \to \Omega : \xi \to \mathcal{L}(\xi)\omega = (A = \mathcal{L}(\xi)\alpha, B = \mathcal{L}(\xi)\beta)$ and the first order involutive system:

The only Vessiot structure equation is $d\alpha = c\beta$ where now d is the exterior derivative and the only Vessiot structure constant is $c = cst$ [5]. We let the reader check that this system is formally integrable (FI), that is no new PD equation can be obtained by differentiating these three equations in any way, if and only if this condition is satisfied. We have the differential sequence:

$$0 \to \Theta \to \xi \overset{\mathcal{D}}{\longrightarrow} \eta \overset{\mathcal{D}_1}{\longrightarrow} \zeta \to 0$$

with $\eta = (A_1, A_2, B)$ and $\zeta \in \wedge^2 T^*$ and we may look for the adjoint differential sequence. Multiplying (A_1, A_2, B) respectively by (μ^1, μ^2, μ^3) and integrating by parts, we obtain the adjoint operator $ad(\mathcal{D})$ in the form:

$$-\alpha_1(\partial_1\mu^1 + \partial_2\mu^2) - \beta(\partial_1\mu^3 - c\mu^2) = \nu^1, \ -\alpha_2(\partial_1\mu^1 + \partial_2\mu^2) - \beta(\partial_2\mu^3 + c\mu^1) = \nu^2$$

Then, multiplying $\zeta = \dot{\partial}_1 A_2 - \dot{\partial}_2 A_1 - cB$ by λ , we obtain $ad(\mathcal{D}_1)$ as:

$$\partial_2\lambda = \mu^1, \ -\partial_1\lambda = \mu^2, \ -c\lambda = \mu^3$$

and obtain the adjoint differential sequence: $\nu \leftarrow \mu \leftarrow \lambda$.

in which $ad(\mathcal{D})$ may not generate the compatibility conditions (CC) of $ad(\mathcal{D}_1)$

We have therefore to consider the two cases [4]:

- $c = 0$: With $\alpha = dx^1, \beta = dx^1 \wedge dx^2$, we have $d\alpha = 0$ and $ad(\mathcal{D}_1)$ is not injective with kernel $\lambda = cst$ but $ad(\mathcal{D})$ is surjective. The Lie pseudogroup Γ is made by transfomations $y^1 = x^1 + a, y^2 = x^2 + f(x^1)$ with $a = cst$ and $f(x^1)$ arbitrary.
- $c \neq 0$: Like Vessiot himself, we may choose $\alpha = x^2 dx^1, \beta : dx^1 \wedge dx^2$ and obtain $d\alpha + \beta = 0$, that is $c = -1$. We check that $ad(\mathcal{D}_1)$ is now injective but that $ad(\mathcal{D})$ is still surjective. The Lie pseudogroup Γ is made by transformations $y^{\mathrm{I}} = f(x^1), y'^2 = x^2/(\partial f/\partial x^1)$ with f invertible.

We have thus discovered that the properties of the adjoint sequence largely depend on the Vessiot structure constant c . Finally, if $\alpha = x^2 dx^1 - x^1 dx^2$, no explicit solution may be provided.

Example 1.3: (Contact transformations) With $m = n = 3, q = 1$ and ground differential field $K = \mathbb{Q}(x)$, we may introduce the 1-form $\alpha = dx^1 - x^3 dx^2 \in T^*$ and consider the system of infinitesimal Lie equations defined by $\mathcal{L}(\xi)(\alpha) = \rho(x)\alpha$ after eliminating the factor ρ . However, we notice that α is not invariant by the contact Lie pseudogroup and cannot be considered as an invariant associated geometric object. In fact, it is known that the corresponding geometric object is a 1-form density ω leading to the system of infinitesimal Lie equations in Medolaghi form:

$$\boxed{\Omega_i \equiv (\mathcal{L}(\xi)\omega)_i \equiv \omega_r\partial_i\xi^r - \frac{1}{2}\omega_i\partial_r\xi^r + \xi^r\partial_r\omega_i = 0}$$

and to the only Vessiot structure equation [6]:

$$\boxed{\omega_1(\partial_2\omega_3 - \partial_3\omega_2) + \omega_2(\partial_3\omega_1 - \partial_1\omega_3) + \omega_3(\partial_1\omega_2 - \partial_2\omega_1) = c}$$

with the only structure constant c . We point out the fact that only the condition of constant Riemannian curvature is known today to be similar and we have explained why such a situation has been produced deliberately by the successors of Cartan. In the present contact situation, we may choose $\omega = (1, -x^3, 0)$ and get $c = 1$ but we may also choose $\omega = (1, 0, 0)$ and get $c = 0$, these two choices both bringing an involutive system. Our problem will be now to construct the differential sequences: $\xi \overset{\mathcal{D}}{\longrightarrow} \Omega \overset{\mathcal{D}_1}{\longrightarrow} \zeta$ and its adjoint sequence: $\nu \overset{ad(\mathcal{D})}{\longleftarrow} \mu \overset{ad(\mathcal{D}_1)}{\longleftarrow} \lambda$

Linearizing the only Vessiot structure equation, we get the corresponding CC system $\mathcal{D}_1\Omega = 0$

$$\begin{gathered} \omega_1(\partial_2\Omega_3 - \partial_3\Omega_2) + \omega_2(\partial_3\Omega_1 - \partial_1\Omega_3) + \omega_3(\partial_1\Omega_2 - \partial_2\Omega_1) \\ +(\partial_2\omega_3 - \partial_3\omega_2)\Omega_1 + (\partial_3\omega_1 - \partial_1\omega_3)\Omega_2 + (\partial_1\omega_2 - \partial_2\omega_3)\Omega_3 = 0 \end{gathered}$$

Multiplying on the left by a test function λ and integrating by parts, we get the operator $ad(\mathcal{D}_1)$ in the form:

$$\left\{ \begin{array}{lll} \Omega_1 \to & \omega_3\partial_2\lambda - \omega_2\partial_3\lambda + 2(\partial_2\omega_3 - \partial_3\omega_2)\lambda & = \mu^1 \\ \Omega_2 \to & \omega_1\partial_3\lambda - \omega_3\partial_1\lambda + 2(\partial_3\omega_1 - \partial_1\omega_3)\lambda & = \mu^2 \\ \Omega_3 \to & \omega_2\partial_1\lambda - \omega_1\partial_2\lambda + 2(\partial_1\omega_2 - \partial_2\omega_1)\lambda & = \mu^3 \end{array} \right.$$

We obtain therefore the crucial formula $2c\lambda = \omega_i\mu^i$ showing how the previous sequences are essentially depending on the Vessiot structure constant c .

• Indeed, if $c \neq 0$, then $\mu = 0 \Rightarrow \lambda = 0$ and the operator $ad(\mathcal{D}_1)$ is injective. This is the case when $\omega = (1, -x^3, 0) \Rightarrow c = 1 \Rightarrow \lambda = 0$.

• On the contrary, if $c = 0$, then the operator $ad(\mathcal{D}_1)$ may not be injective as can be seen by choosing $\omega = (1, 0, 0)$. Indeed, in this case we get a kernel defined by $\partial_3\lambda = 0, \partial_2\lambda = 0$.

Finally, unimodular contact transformations are preserving the 1-form $\alpha = dx^1 - x^3dx^2$, thus also the 2-form $\beta = d\alpha = dx^2 \wedge dx^3$ and even the 3-form $\alpha \wedge \beta = dx^1 \wedge dx^2 \wedge dx^3$. The Vessiot structure equations for the geometric object $\omega = (\alpha, \beta)$ are now $d\alpha = c'\beta, d\beta = c''\alpha \wedge \beta$ with $0 = d^2\alpha = c'd\beta = c'c''\alpha \wedge \beta$ and the only Jacobi condition $c'c'' = 0$ because $\alpha \wedge \beta \neq 0$ [6].

Remark 1.4: When ω is a non-degenerate metric, that is $det(\omega) \neq 0$, it is well known since the work of L.P. Eisenhart on Riemannian geometry in 1926 that the first order Killing linear system defined by the $n(n+1)/2$ PD equations $\Omega_{ij} \equiv \omega_{rj}(x)\partial_i\xi^r + \omega_{ir}(x)\partial_j\xi^r + \xi^r\partial_r\omega_{ij}(x) = 0$ for a vector field $\xi = \xi^r\partial_r$, is formally integrable (FI), that is no new first order PD equation can be obtained iff the metric has a constant Riemannian curvature. We have proved in many books [20, 21] and papers that this is the only example of Vessiot equations known today and have explained in [4] that the reason for such a poor situation is related to a very unpleasant "Mathematical Affair" having to do with the differential Galois theory and involving the best french mathematicians of the beginning of the last century, namely H. Poincaré, E. Picard and G. Darboux (Original letters have been given directly to me by Janet because of the personal dedication of my first 1978 GB book and can be found in the library of ENS in Paris while a photocopy can be found in my 1988 GB book [6].

Meanwhile, mixing differential geometry with homological algebra, M. Kashiwara (1970) has created " differential homological algebra", in order to study differential modules by means of double duality and the corresponding extension modules (See [21] for references and Zbl 1079.93001). By chance, unexpected arguments have been introduced by the brothers E. and F. Cosserat (1909) in order to revisit elasticity and by H. Weyl (1918) in order to revisit electromagnetism through a unique differential sequence only depending on the structure of the conformal group. However, while the Cosserat brothers were only using (translations + rotations), Weyl has only been dealing with (dilatation $^+$ elations) as we shall explain [22, 23].

After recalling the negative answer we already provided in 1995 [19], the main purpose of this paper is to use new techniques of group theory in order to revisit the mathematical foundations of general relativity (GR) and gauge theory (GT) that are leading to gravitational waves. We point out the fact that all the diagrams presented can be obtained by means of computer algebra while using recent packages developed by my former PhD student A. Quadrat and his collaborators [24].

The solution of this striking but difficult problem has been announced, as we already said, is a series of lectures given at the Albert Einstein Institute (AEI, Potsdam, October, 23-27, 2017) "General Relativity and Gauge Theory: Beyond the Mirror" (hal-01632085, 09/11/2017). We shall prove in the next section that the study of the Killing operator done in [14] by means of purely technical relativistic tools has in fact nothing to do with GR and can be solved only counting with fingers. We also invite the reader to look at the more applied presentation (arXiv: 2302.06585) [25]. In this second approach, we advise the reader to have a special look at the photo-elastic beam experiment showing the link that may exist between high level mathematical mathematical tools and their phenomenological approach also done by J.C. Maxwell himself. We finally say that a rough sketch of the Spencer operator for systems with constant coefficients has been provided by F.S. Macaulay in 1916 through " Inverse Systems" in [26].

Before going ahead, let us prove that there may be only two types of differential sequences, the Janet sequence introduced by M. Janet in 1920 and the totally different Spencer sequence introduced by D. C. Spencer in 1970 though both only depend on the Spencer operator [27, 28]. Though the mathematical and physical communities still believe that a differential sequence must always be constructed " step by step", that is, starting with $\mathcal{D}\xi = \eta$ with generating CC $\mathcal{D}_1\eta = 0$, one may start anew with $\mathcal{D}_1\eta = \zeta$ with generating CC $\mathcal{D}_2\zeta = 0$ and so on. However, as we shall see, the Poincaré (also called de Rham out of France !) sequence for the exterior derivative may be defined "as a whole", a fact that led people to believe that the central operator for constructing differential sequences is the exterior derivative, a wrong way indeed (See [29] p 185 + 391).

For this, if E is a vector bundle over the base X , we introduce the q jet bundle $J_q(E)$ with sections $\xi_q : (x) \to (\xi^k(x), \xi^k_i(x), \xi^k_{ij}(x), ...)$ transforming like the sections $j_q(\xi) : (x) \to (\xi^k(x), \partial_i\xi^k(x), \partial_{ij}\xi^k_{ij}(x), ...)$. The Spencer operator $d : J_{q+1}(T) \to T^* \otimes J_q(T)$ allows to compare these sections by considering the differences $(\partial_i\xi^k(x) - \xi^k_i(x), \partial_i\xi^k_j(x) - \xi^k_{ij}(x), ...)$ and so on. For any system $R_q \subset J_q(E)$, differentiating once all the given OD or PD equations, we obtain the first prolongation $R_{q+1} \subset J_{q+1}(E)$ equations and the Spencer operator can be extended to an operator:

$$\boxed{d : \wedge^s T^* \otimes R_{q+1} \to \wedge^{s+1} T^* \otimes R_q : (\xi^k_{\mu,I}(x)\, dx^I) \to \left((\partial_i \xi^k_{\mu,I}(x) - \xi^k_{\mu+1_i,I}(x))\, dx^i \wedge dx^I\right)}$$

We use multi-indices $\mu = (\mu_1, ..., \mu_n)$ with $\mu+1_i = (\mu_1, ..., \mu_i+1, ..., \mu_n)$ and $\mid \mu \mid= \mu_1 + ... + \mu_n$ both with standard multi-index notation for exterior forms and one can check that $d \circ d = 0$. The Spencer operator and the exterior derivative are thus interlaced by the above formula. The restriction of d to the terms of upper order defining the symbols $g_{q+r} = R_{q+r} \cap S_{q+r}T^* \otimes E \subset J_{q+r}(E)$ is " minus" the Spencer map $\delta : \wedge^s T^* \otimes g_{q+1} \to \wedge^{s+1} T^* \otimes g_q$ with $\delta \circ \delta = 0$ through the formula $\xi^k_{\mu,I} dx^I \to \xi^k_{\mu+1_i,I} dx^i \wedge dx^I$ because $\xi^k_{\mu+1_i+1_j} dx^i \wedge dx^j = 0$ when $\mid \mu \mid= q, \mid \nu \mid= q+1$ The system R_q is said to be involutive if its symbol g_q is involutive, that is if all the δ sequences are exact, and the projection $\pi^{q+1}_q : R_{q+1} \to R_q$ is an epimorphism. For any system $R_q \subset J_q(E)$, we may define $R^{(s)}_{q+r} \subset R_{q+r} \subset J_{q+r}(E)$ by differentiating $r+s$ times and keeping only the equations of order $q+r$. When R_q is involutive, we may define the Janet bundles F_r, for $r = 0, 1, \dots, n$, by the short exact sequences [20, 29]:

$$\boxed{0 \to \wedge^r T^* \otimes R_q + \delta(\wedge^{r-1} T^* \otimes S_{q+1}T^* \otimes E) \to \wedge^r T^* \times J_q(E) \to F_r \to 0}$$

We may pick up a section of F_r , lift it up to a section of $\wedge^r T^* \otimes J_q(E)$ that we may lift up to a section of $\wedge^r T^* \otimes J_{q+1}(E)$ and apply d in order to get a section of $\wedge^{r+1} T^* \otimes J_q(E)$ that we may project onto a section of F_{r+1} in order to construct an operator $\mathcal{D}_{r+1} : F_r \to F_{r+1}$ generating the CC of $\mathcal{D}_r$ in the canonical linear Janet sequence:

$$\boxed{0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \cdots \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0}$$

If we have two involutive systems $R_q \subset \hat{R}_q \subset J_q(E)$, the Janet sequence for R_q projects onto the Janet sequence for $\hat{R}_q$ and we may define inductively canonical epimorphisms $F_r \to \hat{F}_r \to 0$ for $r = 0, 1, ..., n$ by comparing the previous sequences for R_q and $\hat{R}_q$.

A similar procedure can also be obtained if we define the Spencer bundles C_r for $r = 0, 1, ..., n$ by the short exact sequences [20, 29]:

$$\boxed{0 \to \delta(\wedge^{r-1} T^* \otimes g_{q+1}) \to \wedge^r T^* \otimes R_q \to C_r \to 0}$$

We may pick up a section of C_r , lift it to a section of $\wedge^r T^* \otimes R_q$, lift it up to a section of $\wedge^r T^* \otimes R_{q+1}$ and apply d in order to construct a section of $\wedge^{r+1} \otimes R_q$ that we may project to C_{r+1} in order to construct an operator $D_{r+1} : C_r \to C_{r+1}$ generating the CC of D_r in the canonical linear Spencer sequence which is another completely different resolution of the set Θ of (formal) solutions of R_q .

$$\boxed{0 \longrightarrow \Theta \overset{j_q}{\longrightarrow} C_0 \overset{D_1}{\longrightarrow} C_1 \overset{D_2}{\longrightarrow} C_2 \overset{D_3}{\longrightarrow} \cdots \overset{D_n}{\longrightarrow} C_n \longrightarrow 0}$$

It can be proved that the Spencer sequence for $R_q \subset J_q(E)$ is the Janet sequence for $R_{q+1} \subset J_1(R_q)$ [22]. However, if we have two systems as above, the Spencer sequence for R_q is now contained

into the Spencer sequence for $\hat{R}_q$ and we may construct inductively canonical monomorphisms $0 \to C_r \to \hat{C}_r$ for $r = 0, 1, ..., n$ by comparing the previous sequences for R_q and $\hat{R}_q$.

Defining F_0 and $\Phi = \Phi_0$ by the short exact sequence $0 \to R_q \to J_q(E) \overset{\Phi}{\longrightarrow} F_0 \longrightarrow 0$, all the previous results can be combined in the following essential commutative and exact diagram:

$$\begin{array}{ccccccc}
 0 & & 0 & & 0 & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \wedge^{r-1} T^* \otimes g_{q+1} & \overset{\delta}{\to} & \wedge^r T^* \otimes R_q & \longrightarrow & C_r & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E & \overset{\delta}{\to} & \wedge^r T^* \otimes J_q(E) & \longrightarrow & C_r(E) & \longrightarrow & 0 \\
 \downarrow & & \downarrow \Phi_0 & & \downarrow \Phi_r & & \\
 \wedge^{r-1} T^* \otimes h_1 & \overset{\delta}{\to} & \wedge^r T^* \otimes F_0 & \longrightarrow & F_r & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & &
\end{array}$$

in which $h_1 \subset T^* \otimes F_0$ is defined as the image of $\sigma_1(\Phi)$ in the exact symbol sequence:

$$0 \longrightarrow g_{q+1} \longrightarrow S_{q+1} T^* \otimes E \overset{\sigma_1(\Phi)}{\longrightarrow} T^* \otimes F_0$$

It follows that the Janet bundles can be defined " as a whole" by the short exact sequences:

Considering the first order Killing system $\mathcal{L}(\xi)\omega = 0$, adding its first prolongation $\mathcal{L}(\xi)\gamma = 0$ while using ξ_2 instead of $j_2(\xi)$, we obtain a second order system $R_2 \subset J_2(T)$. When $n = 2$ and ω is the Euclidean or Minkowskian metric, we have a Lie group of isometries with the 3 infinitesimal generators $\{\partial_1, \partial_2, x^1\partial_2 - x^2\partial_1\}$. If we now consider the Weyl group defined by $\mathcal{L}(\xi)\omega = 2A\omega$ and $\mathcal{L}(\xi)\gamma = 0$, we have to add the only dilatation $x^1\partial_1 + x^2\partial_2$. As for the conformal system $\hat{R}_2 \subset J_2(T)$ defined by $(\mathcal{L}(\xi)\gamma)^k_{ij} = \delta^k_i A_j + \delta^k_j A_i - \omega_{ij}\omega^{kr}A_r$ according to [18], we have to add the two elations $\theta^1 = \frac{1}{2}((x^1)^2 + (x^2)^2)\partial_1 + x^1x^2\partial_2$ and $\theta^2 = \frac{1}{2}((x^1)^2 + (x^2)^2)\partial_2 + x^1x^2\partial_1$. As we have $g_3 = 0, \tilde{g}_3 = 0, \hat{g}_3 = 0$, we have the strict inclusions $R_3 \subset \tilde{R}_3 \subset \hat{R}_3 \subset J_3(T)$ of involutive systems with respective dimensions $3 < 4 < 6 < 20$. Collecting these results, we get the following commutative fundamental diagram I where the upper down arrows are monomorphisms while the lower down arrows are epimorphisms Φ_0, Φ_1, Φ_2 obtained by induction [20, 29]:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & \longrightarrow & \hat{\Theta} & \stackrel{j_3}{\longrightarrow} & 6 & \stackrel{D_1}{\longrightarrow} & 12 & \stackrel{D_2}{\longrightarrow} & 6 & \longrightarrow & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & \longrightarrow & \Theta & \stackrel{j_3}{\longrightarrow} & 3 & \stackrel{D_1}{\longrightarrow} & 6 & \stackrel{D_2}{\longrightarrow} & 3 & \longrightarrow & 0 & & Spencer \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & \longrightarrow & 2 & \stackrel{j_3}{\longrightarrow} & 20 & \stackrel{D_1}{\longrightarrow} & 30 & \stackrel{D_2}{\longrightarrow} & 12 & \longrightarrow & 0 & & hybrid \\
 & & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_2 & & & & \\
0 & \longrightarrow & \Theta & \longrightarrow & 2 & \stackrel{D}{\longrightarrow} & 17 & \stackrel{D_1}{\longrightarrow} & 24 & \stackrel{D_2}{\longrightarrow} & 9 & \longrightarrow & 0 & & Janet \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \hat{\Theta} & \longrightarrow & 2 & \stackrel{\hat{D}}{\longrightarrow} & 14 & \stackrel{\hat{D}_1}{\longrightarrow} & 18 & \stackrel{\hat{D}_2}{\longrightarrow} & 6 & \longrightarrow & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & & & & & 0 & & 0 & & 0 & & & &
\end{array}
$$

It follows that "Spencer and Janet play at see-saw", the dimension of each Janet bundle being decreased by the same amount as the dimension of the corresponding Spencer bundle is increased. The Poincaré sequence for the exterior derivative d is $\wedge^0 T^* \stackrel{d}{\longrightarrow} \wedge^1 T^* \stackrel{d}{\longrightarrow} \wedge^2 T^* \longrightarrow 0$ but it is only at the end of the paper that we shall understand the link with Maxwell equations when $n = 4$. In Special relativity, though surprising it may look like, the above example with $n = 2$ fits with Lorentz transformations if one is using the " hyperbolic" notations $s(h(\phi), ch(\phi), th(\phi) = sh(\phi)/ch(\phi)$ with $x^1 = x, x^2 = ct$ and $ds^2 = (dx^1)^2 - (dx^2)^2$. Indeed, setting $th(\phi) = u/c$ and $th(\psi) = v$ among dimensionless quantities, the Lorentz transformation is now described by the formulas: $\bar{x}^1 = ch(\phi)x^1 - sh(\phi)x^2, \bar{x}^2 = -sh(\phi)x^1 + ch(\phi)x^2$. We obtain thus for the composition of speeds $th(\phi + \psi) = ((u/c) + (v/c))/(1 + (u/c)(v/c))$ without the need of any " gedanken experiment" on light signals. A similar result can be obtained with the ordinary " tangent" for the composition of rotations when using the Euclidean metric $(ds)^2 = (dx^1)^2 + (dx^2)^2$ for the plane (x^1, x^2) .

2 MOTIVATING EXAMPLES

In all the following examples we shall study linear systems of ordinary differential (OD) or partial differential (PD) equations with m dependent variables, n independent variables, order q , Lie groups of dimension p , Lie groups of transformations of a manifold X with tangent bundle T , cotangent bundle T^* , made by a Lie group G acting on X with a graph $X \times G \to X \times X : (x, a) \to (x, y = f(x, a))$ or Lie pseudogroups with $m = n$ and geometric objects $(\omega, \gamma, \rho, ...)$ with perturbations $(\Omega, \Gamma, R, ...)$. Vector bundles over X will be denoted by $(E, F, ...)$ and their sections will be denoted by $(\xi, \eta, \zeta, ...)$. Symmetric covariant tensors will be denoted by $S_q T^*$ while r -forms will be sections of $\wedge^r T^*$. Whenever needed, we shall introduce the adjoint vector bundle $ad(E) = \wedge^n T^* \otimes E^*$ in which E^* is obtained from E by inverting the local transition maps, exactly like T^* is obtained from T . The jet bundle of E will be denoted by $J_q(E)$ as usual with an q injective operator $j_q : E \to J_q(E) : \xi \to j_q(\xi) = (\xi, \partial_i \xi, \partial_{ij} \xi, ...)$. Finally, if K is differential field with derivations ∂_i and d_i are commuting formal derivations with $d_i \mid K = \partial_i$, we shall introduce the non-commutative ring $D = K[d_1, ..., d_n]$ of linear differential operators with coefficients in K and we have $d_i a = a d_i + \partial_i a$ in the operator sense.

Example 2.1: With $m = 1, n = 2, q = 2$, let us consider the second order system $R_2 \subset J_2(E)$ written $d_{22}\xi - bx^2 d_1 \xi = \eta^2$, $d_{12}\xi - a d_{11}\xi = \eta^1$ or $\mathcal{D}\xi = \eta$ with two constant parameter (a, b) and ground differential field $K = \mathbb{Q}(a, b)(x)$. Three different situations may exist:

- If $a = 0, b = 0$, the reader will check at once the existence of the single first order CC $d_2\eta^1 - d_1\eta_2 = 0$ as an operator $\mathcal{D}_1 \eta = 0$ but the second order operator $ad(\mathcal{D})$ does not generate the CC of $ad(\mathcal{D}_1)$ which are generated by a single first order CC.
- If $a = 1, b = 0$, the reader will check at once the existence of the single second order CC $d_{22}\eta^1 - (d_{12} - d_{11})\eta^2 = 0$ and that $ad(\mathcal{D})$ does now generate the CC of $ad(\mathcal{D}_1)$.
- It remains to study the case $a = 1, b = 1$ that will bring surprises. Indeed, differentiating once, we obtain the third order system $R_3 \subset J_3(E)$ with corresponding Janet tabular:

$$
\begin{array}{lcl}
d_{222}\xi - x^2 d_{12}\xi - d_1\xi & = & d_2\eta^2 \\
d_{122}\xi - x^2 d_{11}\xi & = & d_1\eta^2 \\
d_{112}\xi - x^2 d_{11}\xi & = & d_1\eta^2 - d_2\eta^1 \\
d_{111}\xi - x^2 d_{11}\xi & = & d_1\eta^2 - d_2\eta^1 - d_1\eta^1 \\
d_{22}\xi - x^2 d_1\xi & = & \eta^2 \\
d_{12}\xi - d_{11}\xi & = & \eta^1
\end{array}
$$

Though the symbol $g_3 = 0$ defined by $\xi_{222} = 0, \xi_{122} = 0, \xi_{112} = 0, \xi_{111} = 0)$ is trivially involutive, this system is not even formally integrable (FI). Indeed, trying all the usual crossed derivatives, we verac $R_2^{(2)} \subset R_2^{(1)} = R_2 \subset J_2(E)$ with respective dimension $3 < 4 = 4 < 6$. After a few tricky substitutions and eliminations, we obtain the totally new second order PD equation:

$$
A \equiv d_{11}\xi = d_{22}\eta^1 - d_{12}\eta^2 + d_{11}\eta^2 - x^2 d_1\eta^1 \in j_2(\eta)
$$

The hard step is to look for generating CC in the form of an operator $\mathcal{D}_1\eta = \zeta$. We obtain the commutative and exact diagram with $dim(R_3) = 4, dim(R_4) = 3$:

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \longrightarrow & S_4T^*\otimes E & \longrightarrow & S_2T^*\otimes F_0 & \rightarrow & h_2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R_4 & \longrightarrow & J_4(E) & \longrightarrow & J_2(F_0) & \rightarrow & Q_2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_3 & \rightarrow & J_3(E) & \rightarrow & J_1(F_0) & \rightarrow & Q_1 & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

and the long exact connecting sequence: $0 \longrightarrow R_4 \longrightarrow R_3 \longrightarrow h_2 \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow 0$. It follows that $Q_1 \simeq Q_2 = 0$ because $dim(h_2) = 1$ and there cannot exist any first or second order CC. We may start afresh with the new system $R'_2 = R_2^{(2)} \subset R_2 \subset J_2(E)$ which is involutive with symbol $g'_2 = 0$.

$$
\begin{array}{rcl}
d_{22}\xi - x^2 d_1\xi & = & \eta^2 \\
d_{12}\xi & = & A + \eta^1 \\
d_{11}\xi & = & A
\end{array}
\qquad
\begin{array}{|cc|}
\hline
1 & 2 \\
1 & \bullet \\
1 & \bullet \\
\hline
\end{array}
$$

We obtain therefore at once the Fundamental Diagram I for R'_2 :

$$
\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & \Theta & \stackrel{j_2}{\rightarrow} & 3 & \stackrel{d_1}{\rightarrow} & 6 & \stackrel{d_2}{\rightarrow} & 3 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \parallel & & \\
 & & 0 & \rightarrow & 1 & \stackrel{j_2}{\rightarrow} & 6 & \stackrel{d_1}{\rightarrow} & 8 & \stackrel{d_2}{\rightarrow} & 3 & \rightarrow & 0 \\
 & & & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Theta & \rightarrow & 1 & \stackrel{D}{\rightarrow} & 3 & \stackrel{D_1}{\rightarrow} & 2 & \rightarrow & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & & & \\
 & & & & & & 0 & & 0 & & & &
\end{array}
$$

It finally remains to find out the generating CC for the initial second order operator $\mathcal{D}\xi = \eta$ which is neither FI nor involutive. Checking the two dots separately we have the two third order (!) CC:

$$B_1 \equiv d_1A - x^2A - d_1\eta^2 + d_2\eta^1 + d_1\eta^1 = 0, \qquad B_2 \equiv d_2A - x^2A - d_1\eta^2 + d_2\eta^1 = 0$$

$$
\begin{aligned}
d_2A - d_1A - d_1\eta^1 \equiv\ & d_{222}\eta^1 - d_{122}\eta^2 - d_{122}\eta^1 + 2d_{112}\eta^2 \\
& - d_{111}\eta^2 - x^2 d_{12}\eta^1 - x^2 d_{11}\eta^1 = 0
\end{aligned} \tag{1}
$$

Exactly like in [13], we now provide the link existing between these two third order CC and the Spencer operator. Indeed, using R_3 we obtain at once:

$$
\begin{array}{rclcl}
d_1\xi_{11} - \xi_{111} & = & d_1A - x^2A - d_1\eta^2 + d_2\eta^1 + d_1\eta^2 & = & B_1 \\
d_2\xi_{11} - \xi_{112} & = & d_2A - x^2A - d_1\eta^2 + d_2\eta^1 & = & B_2
\end{array}
$$

a result that we shall obtain after one more prolongation by the long exact sequence:

$$0 \rightarrow R_5 \rightarrow J_5(E) \rightarrow J_3(F_0) \rightarrow Q_3 \rightarrow 0 \qquad 0 \rightarrow 3 \rightarrow 21 \rightarrow 20 \rightarrow 2 \rightarrow 0$$

We may thus define $F_1 = Q_3$ with $dim(F_1) = 2$ and define similarly F_2 with $dim(F_2) = 1$ by the long exact sequence:

$$0 \to R_6 \to J_6(E) \to J_4(F_0) \to J_1(F_1) \to F_2 \to 0$$
$$0 \to 3 \to 28 \to 30 \to 6 \to 1 \to 0$$

We have indeed $d_1B_2 - d_2B_1 = 0$ and the exact differential sequence which is not a Janet sequence:

$$\boxed{0 \longrightarrow \Theta \to 1 \underset{2}{\overset{D}{\longrightarrow}} 2 \underset{3}{\overset{D_1}{\longrightarrow}} 2 \underset{1}{\overset{D_2}{\longrightarrow}} 1 \longrightarrow 0}$$

More generally, we have the long exact sequences $\forall r \geq 0$.

$$0 \to 3 \to J_{r+6}(E) \to J_{r+4}(F_0) \to J_{r+1}(F_1) \to J_r(F_2) \to 0 \tag{2}$$

Using finally the basis $\{\theta_\tau(x) \mid 1 \leq \tau \leq 3\} = \{1, x^2, x^1 + \frac{1}{6}(x^2)^3\}$ for the vector space ν over the constants, the Spencer sequence of the Fundamental Diagram I is the tensor product by ν of the Poincaré sequence $\wedge^0 T^* \overset{d}{\to} \wedge^1 T^* \overset{d}{\to} \wedge^2 T^* \longrightarrow 0$ for the exterior derivative when $n = 2$. We shall discover later on that the sitution of the present example with two parameters (a, b) and three different cases is exactly similar to the one that will be provided by the M, S and K metrics with 2 parameters (m, a) and has thus nothing to do with any GR framework.

Example 2.2: With again $m = 1, n = 2, q = 2, K = \mathbb{Q}(x^2)$, the system $R_2 \subset J_2(E)$ defined by $d_{22}\xi - x^2 d_1\xi = 0, d_{12}\xi - \xi = 0$ has the only solution $\xi = 0$ because $R_2^{(3)} = 0$ with $r = 0, s = 3$ and only the central hybrid sequence for j_2 is left:

$$0 \longrightarrow 1 \overset{j_2}{\to} 6 \overset{D_1}{\longrightarrow} 8 \overset{D_2}{\longrightarrow} 3 \longrightarrow 0$$

Example 2.3: (Macaulay) With $m = 1, n = 3, q = 2$ and $P, Q, R \in D = \mathbb{Q}[d_1, d_2, d_3]$, let us consider the linear homogeneous second order system of PD equations $R_2 \subset J_2(E)$ defined by:

$$P\xi \equiv d_{33}\xi = 0,\ Q\xi \equiv d_{23}\xi - d_{11}\xi = 0,\ R\xi \equiv d_{22}\xi = 0$$

but the reader may treat as well the system $(d_{33}\xi - d_{11}\xi = 0, d_{23}\xi = 0, d_{22}\xi - d_{11}\xi = 0)$. Of course, this system is FI because it is homogeneous but we let the reader check through the Janet tabular that g_2 is not involutive though the coordinate system is surely δ -regular because be have full class 3 and full class 2. All the third order jets vanish but $y_{123} - y_{111} = 0$ leading to $dim(g_3) = 1 \Rightarrow dim(R_3) = 8$. Finally $g_4 = 0 \Rightarrow dim(R_4) = 8$ and we could believe that we do not need any PP procedure as R_4 is an ivolutive system because $g_4 = 0$ is trivially involutive and R_2 is finite type like the Killing system. It is important to notice that the knowledge of the first second order operator does not provide any way to obtain the third without passing through the second, contrary to the situation existing in the Janet sequence. Such a procedure is rather “experimental” and must be coherent with a theorem saying that the order of generating CC is one plus the number of prolongations needed to reach a 2-acyclic symbol, that is g_3 must be 2-acyclic [20]. Equivalently the δ -sequence:

$$0 \to \bigwedge^2 T^* \otimes g_3 \overset{\delta}{\to} \bigwedge^3 T^* \otimes g_2 \to 0$$

must be exact. We let the reader prove that the corresponding 3×3 matrix has maximum rank. We recall the dimensions of the following jet bundles:

q	$\to$	0	1	2	3	4	5	6	7
S_qT^*	$\to$	1	3	6	10	15	21	28	36
$J_q(E)$	$\to$	1	4	10	20	35	56	84	120

and the commutative and exact diagram allowing to construct the Spencer bundles $C_r \subset C_r(E)$ and the Janet bundles F_r for $r = 0, 1, ..., n$ with $F_0 = J_q(E)/R_q$, showing that we have indeed with $q = 4$:

$$C_r = \wedge^r T^* \otimes R_q$$

$$C_r(E) = \wedge^r T^* \otimes J_q(E)/\delta(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E$$

$$F_r = \wedge^r T^* \otimes J_q(E)/(\wedge^r T^* \otimes R_q + \delta(\wedge^{r-1} T^* \otimes S_{q+1} T^{\otimes} E))$$

When $R_q \subset J_q(E)$ is involutive, that is formally integrable (FI) with an involutive symbol g_q , then these three differential sequences are formally exact on the jet level and, in the Spencer sequence:

$$0 \longrightarrow \Theta \overset{j_q}{\longrightarrow} C_0 \overset{D_1}{\longrightarrow} C_1 \overset{D_2}{\longrightarrow} ... \overset{D_n}{\longrightarrow} C_n \longrightarrow 0$$

the first order involutive operators $D_1, D_2, ..., D_n$ are induced by the Spencer operator $d : R_{q+1} \to T^* \otimes R_q$ already considered that can be extended to $d : \wedge^r T^* \otimes R_{q+1} \to \wedge^{r+1} \otimes R_q$. A similar condition is also valid for the Janet sequence:

$$0 \longrightarrow \Theta \longrightarrow E \overset{\mathcal{D}}{\longrightarrow} F_0 \overset{\mathcal{D}_1}{\longrightarrow} F_1 \overset{\mathcal{D}_2}{\longrightarrow} ... \overset{\mathcal{D}_n}{\longrightarrow} F_n \longrightarrow 0$$

which can be thus constructed "as a whole" from the previous extension of the Spencer operator (See [29], p 185 and 391 for diagrams missing in [14, 15] !). However, this result is still not known and not even acknowledged today in mathematicl physics, particularly in general relativity which is never using the Spencer δ -cohomology in order to define the Riemann or Bianchi operators. The study of the present Macaulay example will be sufficient in order to justify our comment. First of all, as g_2 is not 2-acyclic and the coeficients are constant, the second order CC are $Qw - Rv = 0,\ Ru - Pw = 0,\ Pv - Qu = 0$ and the simplest resolution is thus:

$$0 \longrightarrow \Theta \longrightarrow 1 \underset{2}{\overset{\mathcal{D}}{\longrightarrow}} 3 \underset{2}{\overset{\mathcal{D}_1}{\longrightarrow}} 3 \underset{2}{\overset{\mathcal{D}_2}{\longrightarrow}} 1 \longrightarrow 0$$

Secondly, as the first prolongation of R_2 becoming involutive is R_4 because $g_4 = 0$, an idea could be to start with the system $R_3 \subset J_3(E)$ but we have proved in ([32], Example 3.14, p 119 to 126) that the simplest formally exact sequence that could be formally exact is quite far from being a Janet sequence as it is:

$$0 \longrightarrow \Theta \longrightarrow 1 \underset{3}{\longrightarrow} 12 \underset{1}{\longrightarrow} 21 \underset{2}{\longrightarrow} 46 \underset{1}{\longrightarrow} 72 \underset{1}{\longrightarrow} 48 \underset{1}{\longrightarrow} 12 \longrightarrow 0$$

Indeed, the Euler-Poincaré characteristic is $1 - 12 + 21 - 46 + 72 - 48 + 12 = 0$ but we notice that the orders of the successive operators may vary up and down.

It remains to work out the Janet and Spencer sequences in the fundamental diagram I :

$$\begin{array}{ccccccccccccccc}
 & & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \to & \Theta & \overset{j_4}{\longrightarrow} & 8 & \overset{d_1}{\longrightarrow} & 24 & \overset{d_2}{\longrightarrow} & 24 & \overset{d_3}{\longrightarrow} & 8 & \to & 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \to & 1 & \overset{j_4}{\longrightarrow} & 35 & \overset{d_1}{\longrightarrow} & 84 & \overset{d_2}{\longrightarrow} & 70 & \overset{d_3}{\longrightarrow} & 20 & \to & 0 \\
 & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_2 & & \downarrow \Phi_3 & & \\
0 \to & \Theta & \to & 1 & \overset{D}{\longrightarrow} & 27 & \overset{D_1}{\longrightarrow} & 60 & \overset{D_2}{\longrightarrow} & 46 & \overset{D_3}{\longrightarrow} & 12 & \to & 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

As there is no group background, it is nevertheless not evident that the above Spencer sequence is isomorphic to the tensor product of the Poincaré sequence for the exterior derivative by a vector space ν of dimension 8. For this, we notice that each solution is a linear combination of polynomials of degree 3 at most in $\mathbb{Q}[x^1, x^2, x^3]$. After tricky substitutions, we just need to choose the basis:

$$\theta_\tau \mid 1 \leq \tau \leq 8 = \{1, x^1, x^2, x^3, x^1x^2, x^1x^3, \frac{1}{2}(x^1)^2 + x^2x^3, \frac{1}{6}(x^1)^3 + x^1x^2x^3\}$$

All the operators are of order 1 but j_4 and $\mathcal{D}$ which are of order 4.

Example 2.4:(Contact transformations revisited) Whith $m = 3, n = 3$, let $\alpha = dx^1 - x^3dx^2 \in T^*$ be the so-called contact 1-form. The Lie pseudogroup of contact transformations $\Gamma = \{y = f(x) \mid dy^1 - y^3dy^2 = a(x)(dx^1 - x^3dx^2)\}$ preserves α up to a factor $a(x)$. Eliminating this factor among the three infinitesimal Lie equations, we obtain two PD equations but this system is neither involutive nor even FI. The system of infinitesimal Lie equations defining the infinitesimal contact transformations $\Theta \subset T$ is obtained by eliminating the factor $\rho(x)$ in the equations $\mathcal{L}(\xi)\alpha = \rho\alpha$ where $\mathcal{L}$ is the standard Lie derivative [26]. This system is thus only generated by η^1 and η^2 below but is not involutive and one has to introduce η^3 in order to obtain the following involutive system $R_1 \subset J_1(T)$ with two equations of class 3 and one equation of class 2, a result leading to

$dim(g_1) = 6$, $dim(g_2) = 10$ and $dim(g_3) = 15$.

$$\begin{array}{rcl|ccc|}
d_3\xi^3 + d_2\xi^2 + 2x^3d_1\xi^2 - d_1\xi^1 & = & \eta^3 & & & \\
 & & & 1 & 2 & 3 \\
d_3\xi^1 - x^3d_3\xi^2 & = & \eta^2 & 1 & 2 & 3 \\
 & & & 1 & 2 & \bullet \\
d_2\xi^1 - x^3d_2\xi^2 + x^3d_1\xi^1 - (x^3)^2d_1\xi^2 - \xi^3 & = & \eta^1 & & &
\end{array}$$

We have one first order CC:

$$d_3\eta^1 - d_2\eta^2 - x^3d_1\eta^2 + \eta^3 = \zeta$$

showing the origin of η^3 and finally get the Janet sequence:

$$\left| \ 0 \to \Theta \to 3 \overset{\mathcal{D}}{\longrightarrow} 3 \overset{\mathcal{D}_1}{\longrightarrow} 1 \to 0 \ \right|$$

Let us now consider the new pseudogroup $\{\Gamma' \subset \Gamma \mid dy^1 - y^3dy^2 = dx^1 - x^3dx^1\}$ preserving the contact form α , thus also $\beta = d\alpha = dx^2 \wedge dx^3$ and $\gamma = \alpha \wedge \beta = dx^1 \wedge dx^2 \wedge dx^3$ which is the so-called volume form. The corresponding infinitesimal Lie equations are successively:

$$d_1\xi^1 - x^3 d_1\xi^2 0,\ d_2\xi^1 - x^3 d_2\xi^2 - \xi^3 = 0,\ d_3\xi^1 - x^3 d_3\xi^2 = 0$$

$$d_1\xi^2 = 0,\ d_1\xi^3 = 0,\ d_2\xi^2 + d_3\xi^3 = 0$$

$$d_1\xi^1 + d_2\xi^2 + d_3\xi^3 = 0$$

Doing the permutation (123)(312) , we obtain the first order system $R_1' \subset J_1(T)$:

$$\left\{\begin{array}{rcl} d_3\xi^3 & = & 0 \\ d_3\xi^2 & = & 0 \\ d_3\xi^1 & = & 0 \\ d_2\xi^3 + d_1\xi^2 & = & 0 \\ d_2\xi^1 - x^2 d_2\xi^2 & = & 0 \\ d_1\xi^1 - x^2 d_1\xi^2 - \xi^3 & = & 0 \end{array}\right. \quad \left|\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \end{array}\right|$$

and we have the strict inclusions $R_1' \subset R_1 \subset J_1(T)$ with $6 < 9 < 12$. We have the new Janet sequence with $3 - 6 + 4 - 1 = 0$:

$$0 \rightarrow \Theta' \rightarrow 3 \xrightarrow{\mathcal{D}'} 6 \xrightarrow{\mathcal{D}_1'} 4 \xrightarrow{\mathcal{D}_2'} 1 \rightarrow 0$$

Contrary to other examples, in the present situation it is easy to work out the Janet sequences but much more difficult to work out the corresponding Spencer sequences. In fact, we obtain the following Fundamental diagram I in which the central hybrid sequence, which is at the same time a Janet sequence for $j_1 i$ and a Spencer sequence for $J_2(T) \subset J_1(J_1(T))$, is the same for $\Theta' \subset \Theta$ and all the operators involved are first order:

$$\begin{array}{ccccccccccccc} & & & & & & 0 & & 0 & & 0 & & 0 & & \\ & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \Theta' & \xrightarrow{j_1} & 6 & \xrightarrow{d_1} & 14 & \xrightarrow{d_2} & 11 & \xrightarrow{d_3} & 3 & \longrightarrow & 0 \\ & & & & & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ & & 0 & \longrightarrow & 3 & \xrightarrow{j_1} & 12 & \xrightarrow{d_1} & 18 & \xrightarrow{d_2} & 12 & \xrightarrow{d_3} & 3 & \rightarrow & 0 \\ & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Theta' & \longrightarrow & 3 & \xrightarrow{D'} & 6 & \xrightarrow{D_1'} & 4 & \xrightarrow{D_2'} & 1 & \rightarrow & 0 & & \\ & & & & & & \downarrow & & \downarrow & & \downarrow & & & & \\ & & & & & & 0 & & 0 & & 0 & & & & \end{array}$$

In actual practice, when g_q is involutive, its knowledge allows to determines the Hilbert function $dim(g_{q+r})$ and finally $dim(R_{q+r})$. Then $dim(C_r)$ may be computed by induction on r . In this example with $q = 1$, one has the commutative and exact diagram with $dim(C_0) = dim(R_1) = 9 \Rightarrow dim(C_1) = 17 \Rightarrow dim(C_2) = 12$ and so on:

$$\begin{array}{ccccccccccc} 0 & \rightarrow & R_{q+2} & \rightarrow & J_2(R_q) & \rightarrow & J_1(C_1) & \rightarrow & C_2 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & R_{q+1} & \rightarrow & J_1(R_q) & \rightarrow & C_1 & \rightarrow & 0 & & \end{array}$$

Unhappily, Spencer and collaborators, being unaware on one side of the computer algebra works of Janet and Gröbner or of the work of Vessiot on geometric objects, were unable to compute explicit examples. In particular, anybody can check that the examples presented in the introduction of [27] have no relation at all with the remaining of the book.

It is thus sometimes easier to exhibit the Janet sequences than the Spencer sequences. By chance the situation will be opposite when we shall deal with general relativity and black holes.

Example 2.5: (Bose conjecture) Last but not least, the following example is the most difficult of this paper as it is exhibiting a mixture of second and third order CC [4]. With $m = n = 3$, the trivial differential field $K = \mathbb{Q}$ and the ring $D = K[d_1, d_2, d_3]$ of differential operators, let us consider the D module M defined by the second order operator:

$$\mathcal{D} : (\xi^1, \xi^2, \xi^3) \rightarrow (\xi^3_{33} - \xi^1_1 = \eta^1, \xi^3_{23} - \xi^2_1 - \xi^3 = \eta^2)$$

A first question is to determine its torsion submodule $t(M)$ or, equivalently, to know whether $\mathcal{D}$ can be parametrized by a certain operator $\mathcal{D}_{-1}$ through a certain number of arbitrary potential functions ϕ . Using double duality, we may construct the adjoint operator $ad(\mathcal{D})$ by multiplying these equations respectively by (λ^1,λ^2) , adding and integrating by parts in order to get successively:

$$ad(\mathcal{D}) : (\lambda^1,\lambda^2) \rightarrow (d_1\lambda^1 = \mu^1, d_1\lambda^2 = \mu^2, d_{33}\lambda^1 + d_{23}\lambda^2 - \lambda^2 = \mu^3)$$
$$ad(\mathcal{D}_{-1}) : (\mu^1,\mu^2,\mu^3) \rightarrow d_{33}\mu^1 + d_{23}\mu^2 - d_1\mu^3 - \mu^2 = \nu$$
$$\mathcal{D}_{-1} : \phi \rightarrow (d_{33}\phi = \xi^1, d_{23}\phi - \phi = \xi^2, d_1\phi = \xi^3)$$

There is a new second order CC for $\mathcal{D}_{-1}$, namely $\eta^3 \equiv d_{33}\xi^2 - (d_{23}-1)\xi^1 = 0$ and we check at once that $d_1\eta^3 = 0$, that is $t(M)$ is generated by η^3 because we have:

$$(d_{23}-1)\eta^1 - d_{33}\eta^2 = (\xi^2_{133} + \xi^3_{33} - \xi^1_{123}) - (\xi^3_{33} - \xi^1_1) = \xi^2_{133} - \xi^1_{123} + \xi^1_1 = d_1\eta^3$$

It follows that $M' = M/t(M) \simeq D$ is a free, thus torsion-free and even projective module defined by $\mathcal{D}'$ which is parametrized by the injective operator $\mathcal{D}_{-1}$:

$$\mathcal{D}' : (\xi^1,\xi^2,\xi^3)(\xi^3_{33} - \xi^1_1 = \eta^1, \xi^3_{23} - \xi^2_1 - \xi^3 = \eta^2, \xi^2_{33} - \xi^1_{23} + \xi^1 = \eta^3)$$

Indeed, we have an isomorphism $M \simeq t(M) \oplus M'$ because the following splitting :

$$d_{33}\phi = \xi^1,\ d_{23}\phi - \phi = \xi^2 \ \Rightarrow\ d_3\phi = d_2\,\xi^1 - d_3\xi^2 \ \Rightarrow\ \phi = d_{22}\xi^1 - d_{23}\xi^2 - \xi^2$$

It is not at all evident that M' can be defined by only two differentially independent PD equations, the first one for (ξ^1,ξ^2) while the second is providing ξ^3 :

$$\xi^2_{33} - \xi^1_{23} + \xi^1 = 0, \xi^2_{123} - \xi^1_{122} + \xi^2_1 + \xi^3 = 0$$

a result showing that the CC operator $\mathcal{D}''$ of the parametrizing operator $\mathcal{D}_{-1}$ may be generated by one second order CC and one third order CC, a striking result providing the short exact sequence:

$$0 \longrightarrow 1 \xrightarrow{\mathcal{D}_{-1}} 3 \xrightarrow{\mathcal{D}''} 2 \longrightarrow 0$$

Now, we may consider the new system:

$$\left\{\begin{array}{rcl} \xi^3_{33} - \xi^1_1 & = & \eta^1 \\ \xi^2_{33} - \xi^1_{23} + \xi^1 & = & \eta^3 \\ \xi^3_{23} - \xi^2_1 - \xi^3 & = & \eta^2 \\ \xi^2_{13} - \xi^1_{12} + \xi^3_3 & = & d_2\eta^1 - d_3\eta^2 \end{array}\right. \qquad \left|\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & \times & \bullet \end{array}\right|$$

The lack of involutivity provides $d_{22}\eta^1 - d_{23}\eta^2 - \eta^2 = \xi^2_{123} - \xi^1_{122} + \xi^2_1 + \xi^3$.

Conversely,tiv $\xi^2_{13} - \xi^1_{12} + \xi^3_3 = 0$ and thus $\xi^2_{133} - \xi^1_{123} + \xi^3_{33} = 0$ However, we have $d_1\eta^3 = \xi^2_{133} - \xi^1_{123} + \xi^1_1 = 0$ d $\eta^1 = \xi^3_{33} - \xi^1_1 = 0$. Similarly, we obtain $\xi^2_{123} - \xi^1_{122} + \xi^3_{23} = 0$ and obtain by subtraction $\eta^2 = \xi^3_{23} - \xi^2_1 - \xi^3 = 0$.

Remark 2.6: Coming back to the previous motivating examples depending on parameters, we notice that the order of the generating CC may highly depend on the parameters. Such a situation, having nothing to do with physics, is nevertheless quite similar to that of the first order Killing system $R_1 \subset J_1(T)$ allowing to define the first order Killing operator $\mathcal{D} : T \rightarrow S_2T^* : \xi \rightarrow \mathcal{L}(\xi)\omega = \Omega$ through the Lie derivative of a non-degenerate metric ω , for example in the case of the M, S, K metrics where a prolongation $R^{(s)}_{q+r} = \pi^{q+r+s}_{q+r}(R_{q+r+s}) \subset R_{q+r}$ must be done. Using the Lie algebra $\vec{\mathcal{G}}$ with dimension 10 for M , 4 for S and 2 for K instead of n , the Spencer sequence is always isomorphic to the tensor product of the Poincaré sequence for the exterior derivative by a Lie algebra that may have a very small dimension as we shall see. Accordingly, we claim:

THE IMPORTANT OBJECT IS NOT THE METRIC BUT ITS GROUP OF INVARIANCE

In particular, the FACT that third order generating CC for the Killing operator may exist has no physical meaning as nobody is knowing a way to select a best candidate among the possible explicit solutions of Einstein equations in vacuum, a mathematical result questioning the origin and existence of black holes as we shall see !. We also notice the fact that the PP procedure is highly depending on the various parameters involved, namely the only parameter m for the S metric which is reduced to the M metric when $m = 0$ while the K metric depends on the two parameters (m,a) and is reduced to the S metric when $a = 0$. We study now this comment.

3 DIFFERENTIAL TOOLS

3.1 From Group Theory to Differential Sequences

Let G be a Lie group with coordinates $(a^\rho) = (a^1,...,a^p)$ acting on a manifold X with a local action map $y = f(x,a)$. According to the second fundamental theorem of Lie, if $\theta_1,...,\theta_p$ are the infinitesimal generators of the effective action of a lie group G on X , then $[\theta_\rho,\theta_\sigma] = c^\tau_{\rho\sigma}\theta_\tau$ where the $c = (c^\tau_{\rho\sigma} = -c^\tau_{\sigma\rho})$ are the structure constants of a Lie algebra of vector felds which can be identified with $\mathcal{G} = T_e(G)$ the tangent space to $\mathcal{G}$ at the identity $e \in G$ by using the action.

More generally, if X is a manifold and G is a lie group (not acting necessarily on X), let us consider gauging maps $a : X \rightarrow G : (x) \rightarrow (a(x))$. If $x + dx$ is a point of X close to x , then the tangent mapping $T(a) : T = T(X) \rightarrow T(G) : dx \rightarrow da = (\partial a/\partial x)dx$ will provide a point $a + da$ close to a

on G . We may bring a back to e on G by acting on a with a^{-1} on the left, that is $b \to a^{-1}b, \forall b \in G$. We get therefore a 1-form $a^{-1}da = A \in T^* \otimes \mathcal{G}$ and the curvature 2-form $F = (\partial_i A^\tau_j(x) - \partial_j A^\tau_i(x) - c^\tau_{\rho\sigma}A^\rho_i(x)A^\sigma_j(x) = F^\tau_{ij}(x)) \in \wedge^2 T^* \otimes \mathcal{G}$ in the nonlinear gauge sequence:

$$\begin{array}{ccccc} X \times G & \longrightarrow & T^* \otimes G & \longrightarrow & \wedge^2 T^* \otimes G \\ a & \longrightarrow & a^{-1}da = A & \longrightarrow & dA - [A,A] = F \end{array}$$

In 1956, at the birth of gauge theory (GT), the above notations were coming from the EM potential A and EM field $dA = F$ of relativistic Maxwell theory. Accordingly, $G = U(1)$ (unit circle in the complex plane) $\longrightarrow \dim(\mathcal{G}) = 1$ was the only possibility to get a 1-form A and a 2-form F with vanishing structure constants $c = 0$.

Choosing now a "close" to e , that is $a(x) = e + t\lambda(x) + \ldots$ and linearizing as usual, we obtain the linear operator $d : \wedge^0 T^* \otimes \mathcal{G} \to \wedge^1 T^* \otimes \mathcal{G} : (\lambda^\tau(x)) \to (\partial_i \lambda^\tau(x))$ and the linear gauge sequence:

$$\wedge^0 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^1 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^2 T^* \otimes \mathcal{G} \xrightarrow{d} \ldots \xrightarrow{d} \wedge^n T^* \otimes \mathcal{G} \longrightarrow 0$$

which is the tensor product by $\mathcal{G}$ of the Poincaré sequence for the exterior derivative.

Considering now a Lagrangian on $T^* \otimes \mathcal{G}$, that is an action $W = \int w(A)dx$ where $dx = dx^1 \wedge \ldots \wedge dx^n$, we may vary it. With $A = a^{-1}da$ we may introduce $\lambda = a^{-1}\delta a \in \mathcal{G} = \wedge^{\cup} T^* \otimes \mathcal{G}$ and get $\delta A^\tau_i = \partial_i \lambda^\tau - c^\tau_{\rho\sigma}A^\rho_i \lambda^\sigma$ ([20], p 180-185). Setting $\partial w/\partial A = \mathcal{A} = (\mathcal{A}^i_\tau) \in \wedge^{n-1} T^* \otimes \mathcal{G}^*$, we obtain the Poincaré equations $\partial_i \mathcal{A}^i_\tau + c^\sigma_{\rho\tau}\mathcal{A}^\rho_i \mathcal{A}^i_\sigma = 0$ as the adjoint of the previous operator (up to sign). Setting now $(\delta a)a^{-1} = \mu \in \mathcal{G}$,we get the adjoint representation $\lambda = a^{-1}((\delta a)a^{-1})a = Ad(a)\mu$ while, introducing B such that $B\mu = A\lambda$, we get the divergence-like equations $\partial_i B^i_\sigma = 0$.

In a different setting, if G acts on X , let $\{\theta_\tau \mid 1 \le \tau \le p = dim(G)\}$ be a basis of infinitesimal generators of the action. If $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index of length $\mid \mu \mid = \mu_1 + \ldots + \mu_n$ and $\mu + 1_i = (\mu_1, .., \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \ldots, \mu_n)$, we may introduce the Lie algebroid $R_q \subset J_q(T)$ with sections defined by $\xi^k_\mu(x) = \lambda^\tau(x)\partial_\mu \theta^k_\tau(x)$ for an arbitrary section $\lambda \in \wedge^0 T^* \otimes \mathcal{G}$ and the trivially involutive operator $j_q : T \to J_q(T) : \theta \to (\partial_\mu \theta, 0 \le |\mu| \le q)$ of order q . We finally obtain the Spencer operator $d : R_{q+1} \to T^* \otimes R_q$ through the chain rule for derivatives [17]:

$$(d\xi_{q+1})^k_{\mu,i}(x) = \partial_i \xi^k_\mu(x) - \xi^k_{\mu+1_i}(x) = \partial_i \lambda^\tau(x)\partial_\mu \theta^k_\tau(x)$$

When q is large enough to have an isomorphism $R_{q+1} \simeq R_q \simeq \Lambda^0 T^* \otimes \mathcal{G}$ and the following linear Spencer sequence in which the operators D_r are induced by d as above:

$$0 \longrightarrow \Theta \xrightarrow{j_q} R_q \xrightarrow{D_1} T^* \otimes R_q \xrightarrow{D_2} \wedge^2 T^* \otimes R_q \xrightarrow{D_3} \ldots \xrightarrow{D_n} \wedge^n T^* \otimes R_q \longrightarrow 0$$

is isomorphic to the linear gauge sequence but with a completely different meaning because G is now acting on X and $\Theta \subset T$ is such that $[\Theta, \Theta] \subset \Theta$. Surprisingly, these results have NEVER been used in the study of the M, S and K metrics [13].

It is not evident, as we already saw, that the projective transformation $y = (ax + b)/(cx + d)$ of the real line with (a, b, c, d) constants is the generic solution of the third order OD equation $(y_{xxx}/y_x) - (3/2)(y_{xx}/y_x)^2 = 0$ and has three infinitesimal generators $(\partial_x, x\partial_x, \frac{1}{2}x^2\partial_x)$ providing the generic solution of the linearized OD equation $\xi_{xxx} = 0$. The situation of contact transformations met in Section 2 and of Lie pseudogroups in general needs new differential geometric tools as follows, not known by physicists because they necessarily involve the Spencer operator [20].

3.2 LIE ALGEBROIDS

If $R_q \subset J_q(E)$ is a system of order q on E , then $R_{q+r} = \rho_r(R_q) = J_r(R_q) \cap J_{q+r}(E) \subset J_r(J_q(E))$ is called the r -prolongation of R_q and is thus the system obtained after differentiating r times. In actual practice, if the system is defined by PDE $\Phi^\tau \equiv a^{\tau\mu}_k(x)\xi^k_\mu = 0$ the first prolongation is defined by adddng the PDE $d_i\Phi^\tau \equiv a^{\tau\mu}_k(x)\xi^k_{\mu+1_i} + \partial_i a^{\tau\mu}_k(x)\xi^k_\mu = 0$ .A Ccordingly, $\xi_q \in R_q \Leftrightarrow a^{\tau\mu}_k(x)\xi^k_\mu(x) = 0$ and $\xi_{q+1} \in R_{q+1} \Leftrightarrow a^{\tau\mu}_k(x)\xi^k_{\mu+1_i}(x) + \partial_i a^{\tau\mu}_k(x)\xi^k_\mu(x) = 0$ as identitties on X or at least over an open subset $U \subset X$. We finally obtain:

$$\partial_i \Phi^\tau - d_i \Phi^\tau \equiv a^{\tau\mu}_k(x)\big(\partial_i \xi^k_\mu(x) - \xi^k_{\mu+1_i}(x)\big) = 0 \Rightarrow d\xi_{q+1} \in T^* \otimes R_q$$

and the Spencer operator restricts to $d : R_{q+1} \to T^* \otimes R_q$.

Definition 3.B.1: We set $R^{(s)}_{q+r} = \pi^{q+r+s}_{q+r}(R_{q+r+s})$ and this system is called the s-prolonged system of order $q + r$. For Example, if we consider the second order Killing system $R_1 \subset J_1(T)$ defined by $x^2\xi^1_1 + \xi^2 = 0, \xi^1_2 = 0$ then $R^{(1)}_1 \subset R_1$ is defined by adding $\xi^1_1 + \xi^2_2 = 0$, even though both systems have the same " solutions".

Definition 3.B.2: The symbol of R_q is the family $g_q = R_q \cap S_q T^* \otimes E$ of vector spaces over X . The symbol g_{q+r} of $\mathcal{R}_{q+r}$ only depends on g_q by a direct prolongation procedure. We may define the vector bundle F_0 over $\mathcal{R}_q$ by the short exact sequence $0 \to R_q \to J_q(E) \to F_0 \to 0$ and we have the exact induced sequence $0 \to g_q \to S_q T^* \otimes E \to F_0$.

When $\mid \mu \mid= q$, we obtain:

$$g_q = \{v^k_\mu \in S_q T^* \otimes E \mid a^{\tau\mu}_k(x)v^k_\mu = 0\}, \mid \mu \mid= q$$

$$\Rightarrow g_{q+r} = \rho_r(g_q) = \{v^k_{\mu+\nu} \in S_{q+r}T^* \otimes E \mid a^{\tau\mu}_k(x)v^k_{\mu+\nu} = 0\},$$

$$\mid \mu \mid= q, \mid \nu \mid= r$$

In general, neither g_q nor g_{q+r} are vector bundles over X as can be seen in the simple example $xy_x - y = 0 \Rightarrow xy_{xx} = 0$.

On $\wedge^s T^*$ we may introduce the usual bases $\{dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_s}\}$ where we have set $I = (i_1 < \ldots < i_s)$. In a purely algebraic setting, one has:

Proposition 3.B.3: There exists a map $\delta : \wedge^s T^* \otimes S_{q+1}T^* \otimes E \to \wedge^{s+1}T^* \otimes S_qT^* \otimes E$ which restricts to $\delta : \wedge^s T^* \otimes g_{q+1} \to \wedge^{s+1}T^* \otimes g_q$ and $\delta^2 = \delta \circ \delta = 0$.

Proof: Let us introduce the map: $\omega = \{\omega^k_\mu = v^k_{\mu,I}dx^I\}$ $(\delta\omega)^k_\mu = dx^i \wedge \omega^k_{\mu+1_i}$ $(\delta^2\omega)^k_\mu = dx^i \wedge dx^j \wedge \omega^k_{\mu+1_i+1_j} = 0$ $a^{\tau\mu}_k(\delta\omega)^k_\mu = dx^i \wedge (a^{\tau\mu}_k\omega^k_{\mu+1_i}) = 0$

The kernel of each δ in the first case is equal to the image of the preceding δ but this may no longer be true in the restricted case and we set:

Definition 3.B.4: Let $B^s_{q+r}(g_q) \subseteq Z^s_{q+r}(g_q)$ and $H^s_{q+r}(g_q) = Z^s_{q+r}(g_q)/B^s_{q+r}(g_q)$ with $H^s(g_q) = H^s_q(g_q)$ be the coboundary space $im(\delta)$, cocycle space $ker(\delta)$ and cohomology space at $\wedge^s T^* \otimes g_{q+r}$ of the restricted g_q is said to be s-acyclic if $H^1_{q+r} = ... = H^s_{q+r} = 0, \forall r \geq 0$. The symbol g_q is said to be of finite type if $g_{q+r} = 0$ for r large enough. In particular, if g_q is involutive and finite type, then $g_q = 0$. Finally, $S_qT^* \otimes E$ is involutive for any $q \geq 0$ if we set $S_0T^* \otimes E = E$.

A first point is provided by the following useful but technical results. As we do not want to provide details about groupoids, we shall introduce a " copy" Y (target) of X (source) and define simply a Lie pseudogroup $\Gamma \subseteq aut(X)$ as a group of transformations solutions of a (in general nonlinear) system $\mathcal{R}_q$, such that, whenever $y = f(x), z = g(y) \in \Gamma$ can be composed, then $z = g \circ f(x) \in \Gamma$, $x = f^{-1}(y) \in \Gamma$ and $y = id(x) = x \in \Gamma$. Setting $y = x + t\xi(x) + ...$ and passing to the limit when $t \to 0$, we may linearize the later system and obtain a (linear) system $R_q \subset J_q(T)$ such that $[\Theta, \Theta] \subset \Theta$. We may use the Frobenius theorem in order to find a generating fundamental set of differential invariants $\{\Phi^\tau(y_q)\}$ up to order q which are such that $\Phi^\tau(\bar{y}_q) = \Phi^\tau(y_q)$ whenever $\bar{y} = g(y) \in \Gamma$. We obtain the Lie form $\Phi^\tau(y_q) = \Phi_\tau(id_q(x)) = \Phi^\tau(j_q(id)(x)) = \omega^\tau(x)$ of $\mathcal{R}_q$.

Of course, in actual practice one must use sections of R_q instead of solutions and we now prove why the use of the Spencer operator becomes crucial for such a purpose. Indeed, we may define:

$$\{j_{q+1}(\xi), j_{q+1}(\eta)\} = j_q([\xi, \eta]), \forall \xi, \eta \in T \qquad (algebraic\ bracket)$$

We may obtain by bilinearity a bracket on $J_q(T)$ extending the bracket on T .

$$[\xi_q, \eta_q] = \{\xi_{q+1}, \eta_{q+1}\} + i(\xi)d\eta_{q+1} - i(\eta)d\xi_{q+1}, \forall \xi_q, \eta_q \in J_q(T) \qquad (differential\ bracket)$$

which does not depend on the respective lifts ξ_{q+1} and η_{q+1} of ξ_q and η_q in $J_{q+1}(T)$. This bracket on sections satisfies the Jacobi identity:

$$[[\xi_q, \eta_q], \zeta_q] + [[\eta_q, \zeta_q], \xi_q] + [[\zeta_q, \xi_q], \eta_q] = 0, \ \forall \xi_q, \eta_q, \zeta_q \in J_q(T) \qquad (Jacobs).$$

and we set [20]:

Definition 3.B.5: We say that a vector subbundle $R_q \subset J_q(T)$ is a system of infinitesimal Lie equations or a Lie algebroid if $[R_q, R_q] \subset R_q$, that is to say $[\xi_q, \eta_q] \in R_q, \forall \xi_q, \eta_q \in R_q$. Such a definition can be tested by means of computer algebra. We shall also say that R_q is transitive if we have the short exact sequence $0 \to R^0_q \to R_q \overset{\pi^q_0}{\to} T \to 0$.

Theorem 3.B.6: The bracket is compatible with prolongations:

$$[R_q, R_q] \subset R_q \quad \Rightarrow \quad [R_{q+r}, R_{q+r}] \subset R_{q+r}, \forall r \geq 0$$

Proof: When $r = 1$, we have $\rho_1(R_q) = R_{q+1} = \{\xi_{q+1} \in J_{q+1}(T) \mid \xi_q \in R_q, d\xi_{q+1} \in T^* \otimes R_q\}$ and we just need to use the following formulas showing how d acts on the various brackets if we set $L(\xi_1)\zeta = [\xi, \zeta] + i(\zeta)d\xi_1$ (See [22] and [26] or [32] for more details):

$$i(\zeta)d\{\xi_{q+1}, \eta_{q+1}\} = \{i(\zeta)d\xi_{q+1}, \eta_q\} + \{\xi_q, i(\zeta)d\eta_{q+1}\}, \quad \forall \zeta \in T$$
$$i(\zeta)d[\xi_{q+1}, \eta_{q+1}] = [i(\zeta)d\xi_{q+1}, \eta_q] + [\xi_q, i(\zeta)d\eta_{q+1}]$$
$$+ i(L(\eta_1)\zeta)d\xi_{q+1} - i(L(\xi_1)\zeta)d\eta_{q+1}$$

The right member of the second formula is a section of R_q whenever $\xi_{q+1}, \eta_{q+1} \in R_{q+1}$. The first formula may be used when R_q is formally integrable.

Corollary 3.B.7: The bracket is compatible with the PP procedure:

$$[R_q, R_q] \subset R_q \quad \Rightarrow \quad [R^{(s)}_{q+r}, R^{(s)}_{q+r}] \subset R^{(s)}_{q+r}, \forall r, s \geq 0$$

Example 3.B.8: When $n = 1$, one has the unusual successive formulas:

$$[\xi, \eta] = \xi\partial_x\eta - \eta\partial_x\xi$$

$$([\xi_1, \eta_1])_x = \xi\partial_x\eta_x - \eta\partial_x\xi_x$$

$$([\xi_2, \eta_2])_{xx} = \xi_x\eta_{xx} - \eta_x\xi_{xx} + \xi\partial_x\eta_{xx} - \eta\partial_x\xi_{xx}$$

$$([\xi_3, \eta_3])_{xxx} = 2\xi_x\eta_{xxx} - 2\eta_x\xi_{xxx} + \xi\partial_x\eta_{xxx} - \eta\partial_x\xi_{xxx}$$

They can be used for linear ($\xi_x = 0$), affine ($\xi_{xx} = 0$) or projective $\xi_{xxx} = 0$)transformations.

Example 3.B.9: With $n = m = 2$ and $q = 1$, let us consider the Lie pseudogroup $\Gamma \subset aut(X)$ of finite transformations $y = f(x)$ such that $y^2dy^1 = x^2dx^1 = \alpha$. Setting $y = x + t\xi(x) + ...$ and linearizing, we get the Lie operator $\mathcal{D}\xi = \mathcal{L}(\xi)\alpha$ where $\mathcal{L}$ is the Lie derivative and the system $R_1 \subset J_1(T)$ of linear infinitesimal Lie equations already met in Definition 3.B.1. Let us choose the two sections $\xi_1, \eta_1 \in J_1(T)$:

$$\xi_1 = \{\xi^1 = 0, \xi^2 = -x^2, \xi^1_1 = 1, \xi^1_2 = 0, \xi^2_1 = 0, \xi^2_2 = 0\} \in R_1$$
$$\eta_1 = \{\eta^1 = x^2, \eta^2 = 0, \eta^1_1 = 0, \eta^1_2 = -x2, \eta^2_1 = 0, \eta^2_2 = 1\} \in R_1$$

The reader may check that $[\xi_1, \eta_1] \in R_1$. However, we have the strict inclusion $R_1^{(1)} \subset R_1$ defined by the additional equation $\xi^1_1 + \xi^2_2 = 0$ and thus $\xi_1, \eta_1 \notin R_1^{(1)}$ though we have $[R_1^{(1)}, R_1^{(1)}] \subset R_1^{(1)}$, a result not evident and lacking in [14] because ξ_1 and η_1 have nothing to do with solutions.

3.3 JANET VERSUS SPENCER

When $R_q \subset J_q(E)$ is an involutive system on E , there is a Fundamental Diagram I with exact columns, where each operator is induced by the extension $d : \wedge^r T^* \otimes J_{q+1}(E) \wedge^{r+1} T^* \otimes J_q(E) : \alpha \otimes \xi_{q+1} \rightarrow d\alpha \otimes \xi_q + (-1)^r \alpha \wedge d\xi_{q+1}$ of the Spencer operator $d : J_{q+1}(E) \rightarrow T^* \otimes J_q(E) : \xi_{q+1} \rightarrow j_1(\xi_q) - \xi_{q+1}$. The upper sequence is the Spencer sequence while the lower sequence is the Janet sequence and the sum $dim(C_r) + dim(F_r) = dim(C_r(E))$ does not depend on the system while the epimorphisms Φ_r are induced inductively by $\Phi = \Phi_0$ [20, 29]:

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & \Theta & \xrightarrow{j_q} & C_0 & \xrightarrow{d_1} & C_1 & \xrightarrow{d_2} \dots \xrightarrow{d_n} & C_n & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & E & \xrightarrow{j_q} & C_0(E) & \xrightarrow{d_1} & C_1(E) & \xrightarrow{d_2} \dots \xrightarrow{d_n} & C_n(E) & \rightarrow & 0 \\
 & & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_n & & \\
0 & \rightarrow & \Theta & \rightarrow & E & \xrightarrow{D} & F_0 & \xrightarrow{D_1} & F_1 & \xrightarrow{D_2} \dots \xrightarrow{D_n} & F_n & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & &
\end{array}$$

For later computations, the sequence $J_{q+1}(E) \xrightarrow{d} T^* \otimes J_q(E) \xrightarrow{d} \wedge^2 T^* \otimes J_{q-1}(E)$ can be described by the images $\partial_i \xi^k_\mu - \xi^k_{\mu+1_i} = X^k_{\mu,i}$ leading to the identities:

$$\partial_i X^k_{\mu,j} - \partial_j X^k_{\mu,i} + X^k_{\mu+1_j,i} - X^k_{\mu+1_i,j} = 0$$

We also recall that the linear Spencer sequence for a Lie group of transformations $G \times X \rightarrow X$, which essentially depends on the action because infinitesimal generators are needed, is locally isomorphic to the linear gauge sequence which does not depend on the action any longer as it is the tensor product of the Poincaré sequence by the Lie algebra $\mathcal{G}$ of G .

The main idea will be to introduce and compare the three Lie groups of transformations: The Poincaré group of transformations with 10 parameters leading to the Killing system R_2 :

$$\Omega_{ij} \equiv (L(\xi_1)\omega)_{ij} \equiv \omega_{rj}(x)\xi^r_i + \omega_{ir}(x)\xi^r_j + \xi^r \partial_r \omega_{ij}(x) = 0$$

$$\Gamma^k_{ij} \equiv (L(\xi_2)\gamma)^k_{ij} \equiv \xi^k_{ij} + \gamma^k_{rj}(x)\xi^r_i + \gamma^k_{ir}(x)\xi^r_j - \gamma^r_{ij}(x)\xi^k_r + \xi^r \partial_r \gamma^k_{ij}(x) = 0$$

The Weyl group of transformations with 11 parameters leading to the Weyl system $\tilde{R}_2$.

$$\Omega_{ij} \equiv \omega_{rj}(x)\xi^r_i + \omega_{ir}(x)\xi^r_j + \xi^r \partial_r \omega_{ij}(x) = 2\Lambda(x)\omega_{ij}(x)$$

$$\Gamma^k_{ij} \equiv \xi^k_{ij} + \gamma^k_{rj}(x)\xi^r_i + \gamma^k_{ir}(x)\xi^r_j - \gamma^r_{ij}(x)\xi^k_r + \xi^r \partial_r \gamma^k_{ij}(x)$$
$$= 0$$

The conformal group of transformations with 15 parameters (4 translations $^{+6}$ rotations + 1 dilatation $^{+4}$ elations) leading to the conformal Killing system $\hat{R}_2$:

$$\Omega_{ij} \equiv \omega_{rj}(x)\xi^r_i + \omega_{ir}(x)\xi^r_j + \xi^r \partial_r \omega_{ij}(x) = 2\Lambda(x)\omega_{ij}(x)$$

$$\Gamma^k_{ij} \equiv \xi^k_{ij} + \gamma^k_{rj}(x)\xi^r_i + \gamma^k_{ir}(x)\xi^r_j - \gamma^r_{ij}(x)\xi^k_r + \xi^r \partial_r \gamma^k_{ij}(x)$$
$$= \delta^k_i \Lambda_j(x) + \delta^k_j \Lambda_i(x) - \omega_{ij}(x)\omega^{kr}(x)\Lambda_r(x)$$

where one has to eliminate the arbitrary function $\Lambda(x)$ and 1-form $\Lambda_i(x)dx^i$ for finding sections, replacing the ordinary Lie derivative $\mathcal{L}(\xi)$ by the formal Lie derivative $L(\xi_q)$, that is replacing $j_q(\xi)$ by ξ_q when needed. When $n=4$, $\hat{R}_2$ is FI but $\hat{g}_2$ is only 2-acyclic while $\hat{g}_3=0$ and we have for the involutive $\hat{R}_3 \simeq \hat{R}_2$ (See [26] for details and counterexamples):

$$
\begin{array}{ccccccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \rightarrow & \hat{\Theta} & \overset{j_3}{\rightarrow} & 15 & \overset{d_1}{\rightarrow} & 60 & \overset{d_2}{\rightarrow} & 90 & \overset{d_3}{\rightarrow} & 60 & \overset{d_4}{\rightarrow} & 15 & \rightarrow 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \rightarrow & 4 & \underset{3}{\overset{j_3}{\rightarrow}} & 140 & \overset{d_1}{\rightarrow} & 420 & \overset{d_2}{\rightarrow} & 504 & \overset{d_3}{\rightarrow} & 280 & \overset{d_4}{\rightarrow} & 60 & \rightarrow 0 \\
 & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_2 & & \downarrow \Phi_3 & & \downarrow \Phi_4 & \\
0 \rightarrow & \hat{\Theta} & \rightarrow & 4 & \underset{3}{\overset{\hat{D}}{\rightarrow}} & 125 & \underset{1}{\overset{\hat{D}_1}{\rightarrow}} & 360 & \underset{1}{\overset{\hat{D}_2}{\rightarrow}} & 414 & \underset{1}{\overset{\hat{D}_3}{\rightarrow}} & 220 & \underset{1}{\overset{\hat{D}_4}{\rightarrow}} & 45 & \rightarrow 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

The top Spencer sequence is the tensor product of the Poincaré sequence by the Lie algebra $\hat{\mathcal{G}}$ of dimension 15 and we may use the inclusions $R_2 \subset \bar{R}_2 \subset \bar{R}_2 \subset J_2(T)$ with $10 < 11 < 15 < 60$. Working by induction, the minimum formally exact resolution on the jet level is:

$$0 \longrightarrow \hat{\Theta} \longrightarrow 4 \longrightarrow 9 \longrightarrow 10 \longrightarrow 9 \longrightarrow 4 \longrightarrow 0$$

with "up and down" orders that must be compared to the above canonical Janet sequence. Of course, finding such numbers can be done by means of computer algebra (arXiv: 1603.05030) or through combinatorics (exercise !) but it will never prove that such a sequence is formally exact as it will involve enormous matrices (up to 840 x 1134 !!!) and cannot be achieved without the help of the Spencer δ -cohomology, still never introduced in GR or conformal geometry [20, 26].

When ω is the M metric, it follows that $\gamma = 0$ and we obtain therefore:

$$X^r_{j,i} - X^r_{ri,j} = (\partial_i \xi^r_{rj} - \xi^r_{rij}) - (\partial_j \xi^r_{ri} - \xi^r_{rij}) = \partial_i \xi^r_{rj} - \partial_j \xi^r_{ri} = n\left(\partial_i \Lambda_j - \partial_j \Lambda_i\right))$$

Dividing by n , we may thus obtain $(F_{ij} = \partial_i \Lambda_j - \partial_j \Lambda_i) \in \wedge^2 T^*$ from $X^r_{rj,i} \in T^* \otimes \hat{g}_2 \subset C_1$ with $dF = 0$ because $\hat{g}_3 = 0$ and thus $\xi^k_{rij} = 0$ in $S_3 T^* \otimes T$.

This result is solving the dream of H . Weyl for exhibiting the conformal origin of electromagnetism in [22]. It is however completely contradicting the standard approach of classical gauge theory based on the group $U(1)$ which is not acting on space-time. In addition, the EM field F is a section of the first Spencer bundle C_1 in the image of D_1 because $(\Lambda, \Lambda_i) \in C_0 = \hat{R}_3 \simeq \hat{R}_2$.

We apply the linearization procedure to the Riemann tensor:

$$\rho = (\rho^k_{l,ij} = \partial_i \gamma^k_{lj} - \partial_j \gamma^k_{li} + \gamma^r_{lj}\gamma^k_{ri} - \gamma^r_{li}\gamma^k_{rj}) \in \Lambda^2 T^* \otimes T^* \otimes T$$

Now, as the linearization $\Gamma \in S_2 T^* \otimes T$ of γ is a tensor, the linearization R of ρ becomes:

$$R^k_{l,ij} = d_i \Gamma^k_{lj} - d_j \Gamma^k_{li} + \gamma^r_{lj}\Gamma^k_{ri} - \gamma^r_{li}\Gamma^k_{rj} + \gamma^k_{ri}\Gamma^r_{lj} - \gamma^k_{rj}\Gamma^r_{li}$$

If ∇ is the covariant derivative, we have $\nabla_r \omega_{ij} = \partial_r \omega_{ij} - \gamma^s_{ir}\omega_{sj} - \gamma^s_{jr}\omega_{is} = 0$ and we may move down the index k . Then, using r as a dumb index, we may consider the first order equations:

$$(L(\xi_1)\rho)_{kl,ij} \equiv R_{kl,ij} \equiv \rho_{rl,ij}\xi^r_k + \rho_{kr,ij}\xi^r_l + \rho_{kl,rj}\xi^r_i + \rho_{kl,ir}\xi^r_j + \xi^r \partial_r \rho_{kl,ij} = 0$$

that can be considered as an infinitesimal first order variation of ρ . As for the Ricci tensor $(\rho_{ij}) \in S_2 T^*$, we notice that $\rho_{ij} = \rho^r_{i,rj} = 0 \Rightarrow R_{ij} \equiv \rho_{rj}\xi^r_i + \rho_{ir}\xi^r_j + \xi^r \partial_r \rho_{ij} = 0$. Using now a cyclic sums on (ijr) , the Bianchi identities are:

$$\beta_{kl,ijr} \equiv \nabla_r \rho_{kl,ij} + \nabla_i \rho_{kl,jr} + \nabla_j \rho_{kl,ri} = 0 \qquad \Leftrightarrow \qquad \beta \equiv \sum_{cycl} (\partial\rho - \gamma\rho) = 0$$

Their linearizations $B_{kl,ijr} = 0$ are sections of the vector bundle F_2 in the short exact sequence:

$$0 \rightarrow F_2 \rightarrow \wedge^3 T^* \otimes g_1 \overset{\delta}{\longrightarrow} \wedge^4 T^* \otimes T \rightarrow 0$$

$$
\begin{aligned}
\dim(F_2) &= \left(\frac{n(n-1)(n-2)}{6}\right)\left(\frac{n(n-1)}{2}\right) - \left(\frac{n(n-1)(n-2)(n-3)}{24}\right) n \\
&= \frac{n^2(n^2-1)(n-2)}{24}
\end{aligned}
$$

because $dim(g_1) = n(n-1)/2$ for any nondegenerate metric, that is $24 - 4 = 20$ when $n = 4$. Such results cannot be even imagined by somebody not aware of the δ -cohomology [30-32].

We have the linearized cyclic sums of covariant derivatives both with their respective symbolic descriptions, not to be confused with the non-linear corresponding ones:

$$\begin{aligned} B_{kl,rij} &\equiv \nabla_r R_{kl,ij} + \nabla_i R_{kl,jr} + \nabla_j R_{kl,ri} \\ &= 0 \quad \mathrm{mod}\ (\Gamma) \\ &\Leftrightarrow \frac{\Sigma}{\mathrm{cycl}}(dR - \gamma R - \rho\Gamma) = 0 \end{aligned}$$

We have thus $\omega \longrightarrow \gamma \longrightarrow \rho \longrightarrow \beta$ and the respective linearizations $\Omega \longrightarrow \Gamma \longrightarrow R \longrightarrow B$.

3.4 Einstein Versus Weyl

It remains to prove that, in this new framework, the Ricci tensor only depends on the symbol $\hat{g}_2 \simeq T^* \subset S_2T^* \otimes T$ deroonation $\hat{R}_2 \subset J_2(T)\ \hat{R}_1 \subset J_1(T)\ \hat{g}_1 \subset T^* \otimes T \quad \omega_{rj}\xi^r_i + \omega_{ir}\xi^r_j - \frac{2}{n}\omega_{ij}\xi^r_r = 0$ ing on any conformal factor. The next commutative diagram covers both situations, taking into account that the equations of both the classical and conformal Killing operator are homogeneous. The purely algebraic Spencer map $\delta : g_{q+1} T^* \otimes g_q$ with symbol $g_q = R_q \cap S_qT^* \otimes E \subset J_q(E)$ is induced by $-d$ and all the sequences are exact by definition but perhaps the left column:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & g_3 & \rightarrow & S_3T^* \otimes T & \rightarrow & S_2T^* \otimes F_0 & \rightarrow F_1 \rightarrow 0 \\
 & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & \\
0 \rightarrow & T^* \otimes g_2 & \rightarrow & T^* \otimes S_2T^* \otimes T & \rightarrow & T^* \otimes T^* \otimes F_0 & \rightarrow 0 \\
 & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & \\
0 \rightarrow & \boxed{\wedge^2T^* \otimes g_1} & \rightarrow & \wedge^2T^* \otimes T^* \otimes T & \rightarrow & \wedge^2T^* \otimes F_0 & \rightarrow 0 \\
 & \downarrow \delta & & \downarrow \delta & & \downarrow & \\
0 \rightarrow & \wedge^3T^* \otimes T & = & \wedge^3T^* \otimes T & \rightarrow & 0 & \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}$$

Chasing in this diagram, we discover that F_1 is just the cohomology $H^2_1(g_1)$ of the first vertical column at $\wedge^2T^* \otimes g_1$ that is the quotient of the cocycle bundle $Z^2_1(g_1)$ kernel of the map $\delta : \wedge^{\cdot 2}T^* \otimes g_1 \wedge^3 T^* \otimes T$ by the coboundary bundle $B^2_1(g_1)$ image of the map $\delta : T^* \otimes g_2 \wedge^2 \otimes g_1$. Needless to say that such mathematical methods have never been used in the stuy of classical or conformal Riemannian geometry and the reason for which the reader will never find other references in the mathematical or physical literature (!).

THEOREM 3.D.1: Introducing the δ -cohomology bundles $Riemann = H^2_1(g_1)$ at $\wedge^2T^* \otimes g_1$ and $Weyl = H^2_1(\hat{g}_1)$ at $\wedge^2T^* \otimes \hat{g}_1$ while taking into account that $g_1 \subset \hat{g}_1$, $g_2 = 0\ \hat{g}_2 \simeq T^*$ and $\hat{g}^3 = 0$, we have the commutative and exact " fundamental diagram II " found in 1983 [30]:

$$
\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & 0 & & Ricci & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & \longrightarrow & Z_1^2(g_1) & \longrightarrow & Riemann & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & T^* \otimes \hat{g}_2 & \overset{\delta}{\longrightarrow} & Z_1^2(\hat{g}_1) & \longrightarrow & Weyl & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_2T^* & \overset{\delta}{\longrightarrow} & T^* \otimes T^* & \overset{\delta}{\longrightarrow} & \wedge^2 T^* & \longrightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & & &
\end{array}
$$

The splitting sequence $0 \to Ricci \to Riemann \to Weyl \to 0$ provides an unusual interpretation of the successive Ricci, Riemann and Weyl tensor bundles. Similarly, the well known splitting sequence $0 \to S_2T^* \overset{\delta}{\to} T^* \otimes T^* \overset{\delta}{\to} \wedge^2 T^* \to 0$ provides an unusual conformal interpretation of the EM field $F = (F_{ij} = -F_{ji}) \in \wedge^2 T^*$ in a coherent way with the tentative of H. Weyl in 1918 [22].

$$
\begin{aligned}
\dim(\text{Riemann}) - \dim(\text{Weyl}) &= \frac{n^2(n^2-1)}{12} - \frac{n(n+1)(n+2)(n-3)}{12} \\
&= \frac{n(n+1)}{2} \\
&= \dim(\text{Ricci})
\end{aligned}
$$

and the Weyl operator is of order 3 when $n = 3$ but of order 2 when $n \geq 4$, a result still neither known but nor even acknowledged today (See arXiv: 1603.05030 for a computer algebra checking by my former PhD student A. Quadrat and [31]). We finally point out that the Bianchi-type operator is of order 2 when $n = 4$ but of order 1 when $n = 5$ (See [26] for more details).

3.5 Applications

Before comparing the Minkowski, Schwarzschild and Kerr metrics as in [13] while comparing to [14], we point out the purely mathematical fact explaining why gravitational waves and black holes cannot exist, not because of an experimental problem of detection but because of a structural problem of equation. Indeed, introducing the Ricci tensor $\rho_{ij} = \rho^r_{i,rj} = \rho_{ji}$ as a linear combination of the second order derivatives of the metric ω , we may linearize it over the locally constant Euclidean or Minkowskian metric in order to obtain a second order linear homogeneous Ricci operator $S_2T^* \to S_2T^* : (\Omega_{ij}) \to (R_{ij})$ with 4 terms along the formula:

$$2R_{ij} = \omega^{rs}(d_{rs}\Omega_{ij} + d_{ij}\Omega_{rs} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri}) = 2R_{ji}$$

which is also valid when $n = 2$ with $2R_{11} = 2R_{22} = d_{11}\Omega_{22} + d_{22}\Omega_{11} - 2d_{12}\Omega_{12}$ and $R_{12} = 0$. Such a result is showing that the important object is not the Riemann tensor or the Einstein tensor but the Ricci tensor according to the previous fundamental diagram II. We obtain therefore:

THEOREM 4.1: The so-called defining equations of the gravitational waves that can be found in any textbook of general relativity like [19], are nothing else that the adjoint of the Ricci operator which is parametrizing the $Cauchy = ad(Killing)$ operator but is going " backwards", that is from right to left as we saw in the Introduction.

Proof. Introducing the test functions $\lambda^{ij} = \lambda^{ji}$, we get:

$$\lambda^{ij}R_{ij} = \omega^{rs}\lambda^{ij}(d_{rs}\Omega_{ij} + d_{ij}\Omega_{rs} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri})$$

Integrating by parts while setting as usual $\Pi = \omega^{ij}d_{ij}$ for the Dalembert operator and exchanging the dumb indices, we get for the adjoint operator:

$$\boxed{\omega^{ij}d_{ij}\lambda^{rs} + \omega^{rs}d_{ij}\lambda^{ij} - \omega^{sj}d_{ij}\lambda^{ri} - \omega^{ri}d_{ij}\lambda^{sj} = \sigma^{rs}}$$

that is exactly the equations of the gravitational waves leading to the Cauchy identities:

$$d_r\sigma^{rs} \equiv \omega^{ij}d_{rij}\lambda^{rs} + \omega^{rs}d_{rij}\lambda^{ij} - \omega^{sj}d_{rij}\lambda^{ri} - \omega^{ri}d_{rij}\lambda^{sj} = 0$$

There is absolutely no need to set $d_i\lambda^{ij}=0$ and we obtain the adjoint sequences when $n=4$.

$$\begin{array}{ccccccc} & & 4 & \xrightarrow{Killing} & 10 & \xrightarrow{Ricci} & 10 \\ 0 & \leftarrow & 4 & \xleftarrow{Cauchy} & 10 & \xleftarrow{ad(Ricci)} & 10 \end{array}$$

without any reference to the Bianchi operator and the induced div operator or even to the Einstein operator. Hence, gravitational waves cannot exist, not for a problem of detection but for a problem of equation, as we have only obtained " a " parametrization of the Cauchy operator which is not even minimum but we can find a minimum one by keeping only λ_{ij} with $i < j$ as in [18]. Indeed, we have already obtained another parametrization with 20 test functions by using the $Beltrami = ad(Riemann)$ operator in the Introduction with $n=4$, a result proving that the Cauchy operator can be parametrized. For the advanced reader, this intrinsic structural fact means that the differential module defined by the Cauchy operator is torsion-free [4]. We finally point out that the Airy, Beltrami or Einstein parametrizations are not responsible for earthquakes as there is no relation between the test functions λ and the deformation Ω of the metric ω , even if $n=2$ as we saw with $\lambda^{ij}R_{ij}=\phi(d_{11}\Omega_{22}+d_{22}\Omega_{11}-2d_{12}\Omega_{12})$ and Airy function ϕ . We also point out that the differential module defined by the Riemann or Ricci operator has a nonzero torsion submodule generated by the 10 components of the Weyl tensor which are separately killed by the Dalembert operator.

In the case of the Killing operator for the Minkowski metric, according to H. Poincaré, the geometrical and adjoint physical long exact differential sequences of operators acting on tensors, giving order of operators and number of components, are exactly the ones we have presented in the Introduction for various dimensions n . The main problem is that the corresponding Killing operators and systems for the M, S or (worst !) K metrics are not involutive and not even formally integrable, a key problem because the Janet sequences cannot be easily constructed. Hence, our main target will be to apply the PP procedure in each case, a difficult task indeed.

As we know that the corresponding Spencer sequence is isomorphic to the tensor product of the Poincaré differential sequence for the exterior derivative by a Lie algebra of dimension $n(n+1)/2$, its adjoint sequence is also exact, the reason for which the physical sequence is also exact, that is each operator is generating the CC of the previous one. In particular, the $Cauchy = ad(Killing)$ operator generates the CC of the $Beltrami = ad(Riemann)$ operator which generates the CC of the $Lanczos = ad(Bianchi)$ operator. We have proved in many recent publications that this result cannot be extended easily to the conformal situation because the acyclicity properties of the symbols highly depend on the dimension n and order q [31]. It thus remains to study the cases of the Schwarzschild and Kerr metrics, a much more delicate problem.

EXAMPLE 4.2: (Kerr metric) We now write the Kerr metric in Boyer-Lindquist coordinates (t,r,θ,ϕ) .

$$\omega = \frac{\rho^2-mr}{\rho^2}\,dt^2 - \frac{\rho^2}{\Delta}\,dr^2 - \rho^2\,d\theta^2 - \frac{2amr\sin^2\theta}{\rho^2}\,dt\,d\phi - \left(r^2+a^2+\frac{mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\,d\phi^2$$

where we have set $\Delta = r^2-mr+a^2$ and $\rho^2=r^2+a^2cos^2(\theta)$ as usual and we recover the S-metric when $a=0$ with $A(r)=1-\frac{m}{r}$:

$$\omega = A(r)dt^2-(1/A(r))dr^2-r^2d\theta^2-r^2sin^2(\theta)d\phi$$

We notice that t or ϕ do not appear in the coefficients of the metric. We shall change the coordinate system in order to confirm these results by using computer algebra. The idea is to use the so-called " rational polynomial " coefficients as follows with $c=cos(\theta)\Rightarrow dc=-sin(\theta)d\theta\Rightarrow dc^2=(1-c^2)d\theta^2$ and set $x^0=t$, $x^1=r$, $x^2=c=cos(\theta)$, $x^3=\phi$). We obtain over the differential field $K=\mathbb{Q}(a,m)(t,r,c,\phi)$:

$$\omega = \frac{\rho^2-mr}{\rho^2}\,dt^2 - \frac{\rho^2}{\Delta}\,dr^2 - \frac{\rho^2}{1-c^2}\,dc^2 - \frac{2amr(1-c^2)}{\rho^2}\,dt\,d\phi - (1-c^2)\left(r^2+a^2+\frac{ma^2r(1-c^2)}{\rho^2}\right)d\phi^2$$

with now $\Delta=r^2-mr+a^2$ and $\rho^2=r^2+a^2c^2$ and we have $det(\omega)=-(r^2+a^2c^2)^2$ in a coherent way with the fact that the S metric that can be written in the new system of coordinates:

$$\omega = A(r)dt^2-\frac{1}{A(r)}dr^2+\frac{r^2}{(1-c^2)}dc^2-r^2(1-c^2)d\phi^2$$

Looking at the symbol $g_1\subset T^*\otimes T$, elementary linear combinatorics allow to prove [18, 22]:

$$\omega_{33}\xi^3_3+\omega_{03}\xi^0_3=0\ mod(\xi), \omega_{33}\xi^3_0+\omega_{00}\xi^0_3=0\ mod(\xi), \omega_{33}\xi^0_0-\omega_{03}\xi^0_3=0\ mod(\xi),$$

Then, multiplying Ω_{22} by ω_{11} , Ω_{11} by ω_{22} and adding, we finally obtain:

$$2(\omega_{11}\omega_{22})(\xi^1_1+\xi^2_2)+\xi\partial(\omega_{11}\omega_{22})=0$$

However, we have also successively:

$$\left\{\begin{array}{lcl} R_{03,03} & \equiv & 2\rho_{03,03}(\xi^0_0+\xi^3_3)+\xi\partial\rho_{03,03}=0 \\ R_{12,12} & \equiv & 2\rho_{12,12}(\xi^1_1+\xi^2_2)+\xi\partial\rho_{12,12}=0 \\ R_{01,23} & \equiv & \rho_{01,23}(\xi^0_0+\xi^1_1+\xi^2_2+\xi^3_3)+\xi\partial\rho_{01,23}=0 \end{array}\right.$$

Now, the coefficients of the metric are rational functions in K and the various geometric objects appearing in $\omega\gamma\rho$ can be obtained through the rules of differential algebra. It is thus possible to obtain the 13 non-zero components of the Riemann tensor for the Kerr metric according to K. R. Koehler in (http://kias.dyndns.org/crg/blackhole.html) by adding factorizations as follows: We shall distinguish the 6 non-vanishing components:

$$\left\{\begin{array}{lcl}
\rho_{01,01} & = & -\frac{mr(2(r^2-mr+a^2)+a^2(1-c^2))(r^2-3a^2c^2)}{2(r^2+a^2c^2)^3(r^2-mr+a^2)} \\
\rho_{02,02} & = & \frac{mr(r^2-mr+a^2+2a^2(1-c^2))(r^2-3a^2c^2)}{2(1-c^2)(r^2+a^2c^2)^3} \\
\rho_{03,03} & = & \frac{mr(1-c^2)(r^2-mr+a^2)(r^2-3a^2c^2)}{2(r^2+a^2c^2)^3} \\
\rho_{12,12} & = & -\frac{mr(r^2-3a^2c^2)}{2(1-c^2)(r^2+a^2c^2)(r^2-mr+a^2)} \\
\rho_{13,13} & = & -\frac{(1-c^2)mr(r^4-2a^2c^2r^2+4a^2r^2-2a^4c^2+3a^4-2a^2mr(1-c^2))(r^2-3a^2c^2)}{2(r^2+a^2c^2)^3(r^2-mr+a^2)} \\
\rho_{23,23} & = & \frac{mr(2r^4-a^2c^2r^2+5a^2r^2-a^4c^2+3a^4-a^2mr(1-c^2))(r^2-3a^2c^2)}{2(r^2+a^2c^2)^3}
\end{array}\right.$$

from the 7 components that are vanishing when $a = 0$.

$$\left\{\begin{array}{lcl}
\rho_{01,23} & = & \frac{amc(2r^2-a^2c^2+3a^2)(3r^2-a^2c^2)}{2(r^2+a^2c^2)^3} \\
\rho_{02,31} & = & -\frac{amc(r^2-2a^2c^2+3a^2)(3r^2-a^2c^2)}{2(r^2+a^2c^2)^3} \\
\rho_{03,12} & = & -\frac{amc(3r^2-a^2c^2)}{2(r^2+a^2c^2)^2} \\
\rho_{02,10} & = & \frac{3a^2mc(3r^2-a^2c^2)}{2(r^2+a^2c^2)^3} \\
\rho_{02,32} & = & \frac{amr(3r^2-mr+3a^2)(r^2-3a^2c^2)}{2(r^2+a^2c^2)^3} \\
\rho_{13,23} & = & -\frac{3a^2mc(1-c^2)(r^2+a^2)(3r^2-a^2c^2)}{2(r^2+a^2c^2)^3} \\
\rho_{01,13} & = & \frac{amr(1-c^2)(3r^2+3a^2-2mr)(r^2-3a^2c^2)}{2(r^2+a^2c^2)^3(r^2-mr+a^2)}
\end{array}\right.$$

We have finally to add the 8 vanishing components:

$$\rho_{01,03} = 0, \quad \rho_{01,12} = 0, \quad \rho_{02,03} = 0, \quad \rho_{02,12} = 0,$$
$$\rho_{03,13} = 0, \quad \rho_{03,23} = 0, \quad \rho_{12,13} = 0, \quad \rho_{12,23} = 0.$$

In fact, as the Riemann tensor has $n^2(n^2-1)/12$ components, that is 20 when $n = 4$, we have to take into account the only identity:

$$\rho_{01,23} + \rho_{02,31} + \rho_{03,12} = 0 \Rightarrow R_{01,23} + R_{02,31} + R_{03,12} = 0$$

We obtain therefore $\xi\partial(\rho_{12,12}/(\omega_{11}\omega_{22}))=0$ but we have also $\xi\partial(\rho_{03,03}\rho_{12,12}/det(\omega))=0$.
The following invariants are obtained successively in a coherent way:

$$\omega_{11}\omega_{22}=\frac{(r^2+a^2c^2)^2}{(1-c^2)(r^2-mr+a^2)}\Rightarrow\mid\rho_{12,12}\mid/(\omega_{11}\omega_{22})=\frac{mr(r^2-3a^2c^2)}{2(r^2+a^2c^2)^3}$$

Also, as $a\in K$, then $\rho_{01,23}$ and $\rho_{02,13}$ can be both divided by a and we get the new invariant:

$$\rho_{01,23}/\rho_{03,12}=\frac{2r^2-a^2c^2+3a^2}{r^2+a^2c^2}$$

These results are leading to $\xi^1=0,\xi^2=0$, thus to $\xi^1_1=0,\xi^2_2=0$ and $\xi^0_0+\xi^3_3=0$ after substitution in the equations defining the first order symbol g_1 of R_1 .

In the case of the S-metric with $a=0$, the previous division has no meaning and we have only $\xi_1=0$ as the only equation of zero order.

Let us now introduce the new equation:

$$R_{01,13}\equiv\rho_{01,13}(\xi^0_0+2\xi^1_1+\xi^3_3)-\rho_{13,13}\xi^3_0-\rho_{01,01}\xi^0_3+(\rho_{01,23}+\rho_{02,13})\xi^2_1=0$$

As we have $\xi^0_0+\xi^3_3=0$ and $\xi^1_1=0$, we obtain therefore a linear equation of the form:

$$\rho_{13,13}\xi^3_0+\rho_{01,01}\xi^0_3-(\rho_{01,23}+\rho_{02,13})\xi^2_1=0$$

Similarly, we have also:

$$R_{01,02}\equiv\rho_{01,02}(2\xi^0_0+\xi^1_1+\xi^2_2)-(\rho_{01,23}+\rho_{02,13})\xi^3_0+\rho_{01,01}\xi^1_2+\rho_{02,02}\xi^2_1=0$$

and we obtain therefore a linear equation of the form:

$$2\rho_{01,02}\xi^0_0-(\rho_{01,23}+\rho_{02,13})\xi^3_0+\rho_{01,01}\xi^1_2+\rho_{02,02}\xi^2_1=0$$

In the case of the S-metric, that is when $a=0$, we obtain respectively $\xi^0_3=0$ and $\xi^1_2=0$ as in [18] because $\xi^0_0\simeq\xi^0_3$ The previous linear system has thus a rank equal to 2 and we obtain therefore because $\xi^3_0\simeq\xi^0_3,\xi^2_1\simeq\xi^1_2$.

$$\xi^0_3=0,\xi^1_2=0,\Leftrightarrow\xi^3_0=0,\xi^2_1=0,\xi^0_0=0,\xi^3_3=0$$

It remains to study the following 4 linear equations, namely [13]:

$$R_{01,03}=0,\ R_{03,23}=0,\ R_{03,13}=0,\ R_{02,03}=0$$

The rank of the previous system with respect to the 4 jet coordinates $(\xi^1_0,\xi^2_0,\xi^1_3,\xi^2_3)$ is equal to 2, for both the S and K-metrics thanks to the two striking identities:

$$R_{03,13}+a(1-c^2)R_{01,03}=0,\quad R_{02,03}+\frac{a}{(r^2+a^2)}R_{03,23}=0$$

Two prolongations only provide 6 additional equations of order one that we provide in the following list which is obtained $mod(j_2(\Omega))$, namely:

$$\xi^1=0,\xi^2=0,\xi^1_2=0,\xi^0_3=0,\xi^1_3+lin(\xi^1_0,\xi^2_0)=0,\xi^2_3+lin(\xi^1_0,\xi^2_0)$$

We have therefore obtained the inclusion of Lie algebroids $R^{(2)}_1\subset R^{(1)}_1=R_1\subset J_1(T)$ with respective dimensions $4<10=10<20$. Using the standard $ker-coker$ long exaxt sequence of prolongations:

$$\boxed{0\rightarrow R_3\rightarrow J_3(T)\rightarrow J_3(S_2T^*)\rightarrow Q_2\rightarrow 0,\quad 0\rightarrow 4\rightarrow 140\rightarrow 150\rightarrow 14\rightarrow 0}$$

we discover that the initial Killing system for the Kerr metric has 14 compatibility conditions of second order contrary to the 20 existing for the Minkowski metric. Such a result has been obtained totally independently of any specific GR technical object like the Teukolski scalars or the Killing-Yano tensors introduced in [14, 15]. However, this system is not involutive because its symbol is finite type but non-zero.

Using one more prolongation, all the sections (care again) vanish but ξ^0 and ξ^3 , a result leading to $dim(R^{(3)}_1)=2$ in a coherent way with the only nonzero Killing vectors $\{\partial_t,\partial_\phi\}$. We have indeed:

$$\xi^1_0=0\ \xi^2_0=0\Big|\ \Leftrightarrow\ \xi^1_3=0,\ \xi^2_3=0\ \Rightarrow\ \xi^0_1=0,\ \xi^0_1=0,\ \xi^0_2=0,\ \xi^3_2=0$$

Taking therefore into account that the metric only depends on ($x^1=r,x^2=cos(\theta)\rangle$ we obtain after three prolongations the inclusions of first order systems:

$$R^{(3)}_1\subset R^{(2)}_1\subset R^{(1)}_1=R_1\subset J_1(T)\Leftrightarrow 2<4<10=10<20$$

Surprisingly and contrary to the situation found for the S metric, we have now an involutive first order system with only solutions ($\xi^0=cst,\xi^1=0,\xi^2=0,\xi^3=cst$) and notice that $R^{(3)}_1$ does not depend any longer on the parameters $(m,a)\in K$. The difficulty is to know what second members must be used along the procedure met for all the motivating examples, in particular we have again identities to zero like $d_0\xi^1-\xi^1_0=0,d_0\xi^2-\xi^2_0=0$.

We finally obtain 14 second order generating CC and their prolongations as we already said but also 6 third order CC coming from the 6 following components of the Spencer operator, namely:

$$d_1\xi^1 - \xi^1_1 = 0, \quad d_2\xi^1 - \xi^1_2 = 0, \quad d_3\xi^1 - \xi^1_3 = 0,$$
$$d_1\xi^2 - \xi^2_1 = 0, \quad d_2\xi^2 - \xi^2_2 = 0, \quad d_3\xi^2 - \xi^2_3 = 0.$$

a result that cannot be even imagined from [14]. Of course, proceeding like in the motivating examples, we must substitute in the right members the values obtained from $j_2(\Omega)$ and set for example $\xi^1_1 = -\frac{1}{2\omega_{11}}\xi^r\partial_r\omega_{11}$ while replacing ξ^1 and ξ^2 by the corresponding linear combinations of the Riemann tensor already obtained for the right members of the two zero order equations. The corresponding Fundamental Diagram I is no longer depending on (m, a) as follows:

$$\begin{array}{ccccccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \rightarrow & \Theta & \stackrel{j_1}{\rightarrow} & 2 & \stackrel{D_1}{\rightarrow} & 8 & \stackrel{D_2}{\rightarrow} & 12 & \stackrel{D_3}{\rightarrow} & 8 & \stackrel{D_4}{\rightarrow} & 2 & \rightarrow 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \rightarrow & 4 & \stackrel{j_1}{\rightarrow} & 20 & \stackrel{D_1}{\rightarrow} & 40 & \stackrel{D_2}{\rightarrow} & 40 & \stackrel{D_3}{\rightarrow} & 20 & \stackrel{D_4}{\rightarrow} & 4 & \rightarrow 0 \\
 & & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \Theta & \rightarrow & 4 & \stackrel{D}{\rightarrow} & 18 & \stackrel{D_1}{\rightarrow} & 32 & \stackrel{D_2}{\rightarrow} & 28 & \stackrel{D_3}{\rightarrow} & 12 & \stackrel{D_4}{\rightarrow} & 2 & \rightarrow 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

with the Euler-Poincaré characteristic $4 - 18 + 32 - 28 + 12 - 2 = 0$. However, the only intrinsic concepts associated with a differential sequence are the " extension modules " that only depend on the Kerr differential module but not on the differential sequence and we repeat once more that: THE ONLY IMPORTANT CONCEPT IS THE GROUP INVOLVED, NOT THE METRIC. Needless to say that the group involved in this case has no physical usefulness.

EXAMPLE 4.3: (Schwarzschild metric) According to what we said, the situation of the Schwarzschild metric is much simpler when $a = 0$ with only six non-zero components of the Riemannn tensor:

$$\begin{array}{|lcl|}
\hline
\rho_{01,01} & = & -\frac{m}{r^2(r-m)} \\
\rho_{02,02} & = & \frac{m}{2r} \\
\rho_{03,03} & = & \frac{m(1-c^2)}{2r} \\
\rho_{12,12} & = & -\frac{m}{2(r-m)} \\
\rho_{13,13} & = & -\frac{m(1-c^2)}{2(r-m)} \\
\rho_{23,23} & = & mr(1-c^2) \\
\hline
\end{array}$$

We have now the inclusions of algebroids:

$$R_1^{(3)} \subset R_1^{(2)} \subset R_1^{(1)} = R_1 \subset J_1(T) 4 < 5 < 10 = 10 < 20$$

The subsystem $R_1^{(2)}$ is def by g the 5 nest rd eutios:

$$|\ \xi^1 = 0, \xi^1_2 = 0, \xi^1_3 = 0, \xi^0_2 = 0, \xi^0_3 = 0$$

while $R_1^{(3)}$ is obtained by adding aain $\xi^1_0 = 0$. We have 15 generating second order CC and 3 third order CC provided by the Spencer operator:

$$d_1\xi^1 - \xi^1_1 = 0, d_2\xi^1 - \xi^1_2 = 0, d_3\xi^1 - \xi^1_3 = 0$$

in which we have to substitute $\xi^1 \in j_2(\Omega)$, $\xi^1_1 \in j_2(\Omega)$, $\xi^1_2 \in j_2(\Omega)$, $\xi^1_3 \in j_2(\Omega)$.

It thus follows from the previous results of this section, which have been obtained without the need of any purely relativistic tool, that black holes cannot exist as they are contradicting [14,15] while showing that the underlying mathematical problem is a purely formal one with no link with GR along the given motivating example. ï»¿

CONCLUSION

When a linear partial differential operator $\mathcal{D}\xi = \eta$ is given, a direct problem is to look for the generating compatibility conditions $\mathcal{D}_1\eta = 0$ that must be satisfied by η . Similarly, if $\mathcal{D}_1\eta = \zeta$ is given, one may look for CC of the form $\mathcal{D}_2\zeta = 0$ and so on. The mathematical community (and we do not speak about the physical community !) is of course aware of such a " step by step" way but is not at all aware of the existence of another "as a whole" procedure allowing to define the various differential operators of the differential sequence thus obtained apart from the very specific situation of the Poincaré (in France !) sequence for the exterior derivative that admits a unique defining formula for each operator separately. The best known case is that of Riemannian geometry and its application to general relativity with the successive Killing, Riemann and Bianchi operators of first, second and first order respectively. In particular, we may ask " Who knows about the Spencer operator and the corresponding Spencer sequence $\ell^{\,n}$ at the heart of this paper.

In the Introduction, we have explained and illustrated through many motivating examples that, when a second order differential operator $\mathcal{D}$ is depending on constant or variable coefficients, its generating compatibility conditions (CC) may be of first, second, third and even sixth or higher order, a result largely depending on the parameters. In the meantime, we have shown that the solution of this problem for a system of order q cannot be obtained without bringing such a system to an involutive form or at least to a formally integrable form of order $q+r$ after differentiating $r+s$ times the equations while keeping only the equations left up to order $q+r$ in such a way that the order of the CC is at most $r+s+1$.

From a very different point of view, the Spencer differential sequence is obtained by bringing any involutive system $R_q \subset J_q(E)$ to a first order involutive system $R_{q+1} \subset J_1(R_q)$ having an isomorphic space of solutions or, with a more precise language, allowing to define a differential module isomorphic to the differential module M defined by the initial system. The quotient of the Spencer sequence for the first order trivially involutive first order system $J_{q+1}(E) \subset J_1(J_q(E))$ by the previous Spence sequence which is induced by the inclusion $R_q \subset J_q(E)$ is the well defined finite length differential Janet sequence introduced by M. Janet as a footnote in 1920 which is thus providing another resolution of the same space of solutions or of the differential module M already defined. According to a very difficult theorem of (differential) homological algebra, the only objects that do not depend of the resolution used are the (differential) extension modules that are measuring the fact that the corresponding dual sequence made by the respective formal adjoint of the operators involved and going thus " backwards" (that is from right to left) may not be exact, that is each operator may not generate the CC of the preceding one. it thus follows that the Spencer and Janet sequences will bring the same formal informations as a whole, even though, in actual practice, we proved that they can be completely different.

It may happen, for example with the Schwarzschild and Kerr metrics, with $q=1, r=0, s=3$, that the final corresponding FI systems will not depend any longer on the parameters involved initially. Accordingly, the only important object to consider is not the metric but its group G of invariance which is used through the fact that the Spencer sequence is the tensor product of the Poincaré sequence by its Lie algebra $\vec{\mathcal{G}}$, the main formal reason for which black holes cannot exist.

We have thus finally proved that the main idea, along the tentative of H. Weyl in 1918, is not to shrink the dimension of this group from 10 down to to 4 or 2 parameters by using the S or K metrics instead of the M metric but, on the contrary, to enlarge the group from 10 up to 11 or 15 parameters by using the Weyl or conformal group instead of the Poincar'e group of space-time while using the adjoint of the respective Spencer sequences [17]. It will follow that the first set of Maxwell equations is obtained by a projection of the second Spencer operator D_2 that can be parametrized by a projection of the first Spencer operator D_1 , a result contradicting the basic assumption of classical gauge theory in which they are induced by D_3 . The main problem today is that, in the minimum resolution of the conformal Killing operator, the generating CC of the second order Weyl operator are also made by a second order operator, a result confirmed by my PhD student A. Quadrat (INRIA) in 2016 (See [24, 26] or arXiv: 1603.05030) but still not acknowledged and showing that conformal geometry but me revisited by using the Spencer δ -cohomology [31].

Finally, in a more general framework, when a Lie group G is acting on a manifold X of dimension n , the Spencer sequence is always locally and formally exact, being isomorphic to the tensor product of the Poincaré sequence by the Lie algebra of G . On the contrary, the corresponding formally exact Janet sequence may have Janet bundles of high dimensions. The last operator $\mathcal{D}_n$ is always surjective while its adjoint is always injective. However, $ad(\mathcal{D}_{n-1})$ may not define all the CC of $ad(\mathcal{D}_n)$ because $ad(D_n)$ may fail to be injective. Applying these methods to the conformal group of transformations when $n=4$, we discovered in the very recent [17] the common conformal origin of the Cauchy, Cosserat, Clausius/Poisson and Maxwell equations. For example the EM field F comes from the composition of epimorphisms ($60 \rightarrow 16 \rightarrow 6 \rightarrow 0$):

$$\boxed{\hat{C}_1 \rightarrow \hat{C}_1/\tilde{C}_1 = (T^* \otimes \hat{R}_2)/(T^* \otimes \tilde{R}_2) \simeq T^* \otimes (\hat{R}_2/\tilde{R}_2) \simeq T^* \otimes \hat{g}_2 \simeq T^* \otimes T^* \stackrel{\delta}{\rightarrow} \wedge^2 T^*}$$

while the EM potential comes from the composition of epimorphisms (1540):

$$\hat{C}_0\hat{C}_0/\tilde{C}_0 \simeq \hat{R}_2/\tilde{R}_2 \simeq \hat{g}_2 \simeq T^*$$

and the parametrization $dA = F$ is induced by D_1 while the Maxwell equation $dF = 0$ is induced by D_2 . Using the short exact splitting sequence $0 \rightarrow S_2T^* \stackrel{\delta}{\longrightarrow} T^* \otimes T^* \stackrel{\delta}{\longrightarrow} \wedge^2 T^* \rightarrow 0$ leading to the isomorphism $T^* \otimes T^* \simeq S_2T^* \oplus \wedge^2 T^* \simeq (R_{ij}) \oplus (F_{ij})$ with $16 = 10 + 6$, that only depends on the elations of the conformal group, along the Fundamental Diagram II showing the common conformal origin of electromagnetism and gravitation, already published in 1983 [30]. More generally, taking the quotient of the Spencer sequence for $\hat{C}_r$ by the Spencer sequence for $\tilde{\Theta}$, we obtain a formally exact sequence with bundles $\hat{C}_r/\tilde{C}_r$ which is isomorphic to the tensor product of the Poincar
'e sequence by a vector space of dimension $dim(\hat{g}_2) = dim(T^*) = 4$ that we can project onto the formally exact Poincaré sequence, with a shift by one step contradicting the use of $U(1)$ in GT. We obtain therefore the following commutative and exact diagram in which we notice that the first vertical short exact sequence on the left is nothing else than the bottom short exact sequence of the previous fundamental diagram II:

$$
\begin{array}{ccccccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 1 & \rightarrow & 10 & \rightarrow & 20 & \rightarrow & 15 & \rightarrow & 4 & \rightarrow & 0 & & gravitation \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \parallel & & & & \\
 & & 4 & \stackrel{d}{\rightarrow} & 16 & \stackrel{d}{\rightarrow} & 24 & \stackrel{d}{\rightarrow} & 16 & \stackrel{d}{\rightarrow} & 4 & \rightarrow & 0 & & \\
 & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
1 & \stackrel{d}{\rightarrow} & 4 & \stackrel{d}{\rightarrow} & 6 & \stackrel{d}{\rightarrow} & 4 & \stackrel{d}{\rightarrow} & 1 & \rightarrow & 0 & & & & electromagnetism \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & & & \\
 & & & & 0 & & 0 & & 0 & & & & & &
\end{array}
$$

A snake chase in this diagram proves that gravitation has to do with the conformal factor ϕ when $F = 0$ as the first operator on the upper left is just the composition $D_1 \circ d : \phi \rightarrow \partial_i \phi \rightarrow \partial_{ij} \phi$ according to the fundamental diagram II. Also, using the dual summation $\rho(x)(\partial_4 \xi^4 - \xi^4_4) + g^i(x)(\partial_i A - A_i) + g^{ij}(x)(\partial_i A_j - 0)$ with $\xi^k_{ijr} = 0$ and integrating by parts, we get the induction equations:

$$\xi^4 \partial_4 \rho = 0, \xi^4_4 = A\partial_i g^i + \rho = 0, \xi^4_{4i} = A_i \rightarrow \partial_r g^{ri} + g^i = 0$$

leading to the Poisson equation for ϕ when $g^{ij} = \phi \omega^{ij}$ and thus to a new conformal scheme for electromagnetism and gravitation. It is also a reason for which elations are also sometimes called and confused with “accelerations” because and accelerometer on an inertial platform in a rocket can only measure the difference $\partial_4 \xi^k_4 - \xi^k_{44}$ with $k = 1, 2, 3$ between the acceleration (time derivative of the speed as a rotation of space-time) and the gravitational field, while one has $\xi^k_{44} = \omega^{kr} A_r$ and the ditive ruleof secn rderts

$$\boxed{\bar{f} = g \circ f \Rightarrow \bar{f}^k_{44} = f^k_{44} + g^k_{44}}$$

holds whenever $f_1 = g_1 = id_1 = j_1(id)$ for the one jet of the identity transformation $y = x$ (Compare [25] to [33]). However, no one of these results could have been even imagined by Weyl because the Riemann operator must be replaced by the Weyl operator while the Bianchi operator must be replaced by a second order CC operator, a result still not known today in the conformal framework [33, 34].

In the spirit of the Cosserat brothers on elasticity in 1909 and H. Weyl on electromagnetism in 1918, we have tried to explain and illustrate why the structure of physical theories must depend on a Lie group of transformations considered as a Lie pseudogroup. The main idea is then to introduce the corresponding Janet (1920) or Spencer (1970) differential sequences both with their respective adjoint differential sequences needed for variational calculus [28, 35]. In a coherent way with the dream of Weyl, the most striking result is the “shift” that must be used in the physical interpretation of the sequences obtained, in the sense that the EM field as a 2-form is coming from the first Spencer bundle and not from the second (See [16] for other examples). These two sequences may be sometimes quite different though always homologically equivalent and we have tried to illustrate them through the study of gravitational waves and black holes. As both of them are largely unknown, no other reference can be found and only the future will decide about the best one that must be used !.

APPENDIX

After 1950 the introduction of homological algebra in pure mathematics has been more than an evolution but rather quite a revolution in algebraic geometry. Following the wish of one of the reviewers that we thank, our aim in this appendix is to explain that differential homological algebra, namely the way to add the word ” differential ” in front of ALL the concepts already introduced (module, resolution, duality, extension modules, ..) will bring quite a revolution in physics. The main problem is that a full account should need a whole paper [35] or even a whole book [21] and that we have seen in the core of this paper that the differential geometric framework is already difficult while the geometric part is somehow even more difficult. By chance, in a more general framework, we have proved that the controllability of a control system is a structural property of the control system, not depending on the choice of inputs and outputs among the control variables and can be described in terms of vanishing of extension modules [4]. Roughly, it means that a control system is controllable if and only if the system can be parametrized or, equivalently, if there does not exist any observable quantity which is solution of an autonomous system for itself, a more physical way to say that the differential module defined by the system is torsion-free [35]. Accordingly we shall explain through two examples how to pass ” beyond the mirror”.

First of all, we sketch the idea hidden beyond ” differential extension modules” by using only the language of operators along the content of this paper. For this, we notice that, if a given linear differential operator $\mathcal{D}_1 \eta = \zeta$ is give, a ” direct problem ” is to look for compatibility conditions (CC) in the form of a linear operator $\mathcal{D}_2 \zeta = 0$ and so on. We hope that the reader is now convinced that it may already be a difficult problem to provide ” a prior” any idea of the order of the CC operator as we have provided examples in which the order is ranging from 1 up to 6. However, conversely, if a system described by $\mathcal{D}_1 \eta = 0$, a much more difficult ” inverse problem ” is to find a parametrization, that is an operator $\mathcal{D}\xi = \eta$ having exactly such CC. Now, using the well known properties of the adjoint procedure $\mathcal{D}ad(\mathcal{D})$ with $ad(ad(\mathcal{D})) = \mathcal{D}$, if $\mathcal{D}\xi = \eta$ has generating CC $\mathcal{D}_1 \eta = 0$, then $\mathcal{D}_1 \circ \mathcal{D} = 0 \Rightarrow ad(\mathcal{D}) \circ ad(\mathcal{D}_1) = 0$ but $ad(\mathcal{D})$ may not generate all the CC of $ad(\mathcal{D}_1)$ while $ad(D_1)$ may not generate all the CC of $ad(D_2)$ and

$ad(\mathcal{D}_2)$ may not generate all the CC of $ad(\mathcal{D}_3)$ and so on, as can be seen in the motivating examples. The introduction of extension modules is a way to measure such a gap.

Starting with the system, we sketch on an example the five steps of the differential double duality test must be used in order to know whether this system can be parametrized or not:

STEP 1: Write down the system $\mathcal{D}_1\eta = \zeta : d_{12}\eta^1 + d_{22}\eta^2 = \zeta$.

STEP 2: Write down the adjoint operator $\cdots$ backwards: multiplying on the left by a test function λ and integrating by parts, we get

$$\boxed{\eta^1 \rightarrow d_{12}\lambda = \mu^1, \eta^2 \rightarrow d_{22}\lambda = \mu^2}$$

STEP 3: As any operator is the adjoint of its adjoint, find its CC as an operator $ad(\mathcal{D})\mu = \nu \; d_2\mu^1 - d_1\mu^2 = \nu$.

STEP 4: Write down its adjoint as an operator $\mathcal{D}\xi = \eta$. $\mu^1 \rightarrow -d_2\xi = \eta^1, \mu^2 \rightarrow d_1\xi = \eta^2$

Step 5: Find its CC as an operator $\mathcal{D}_1'\eta = 0$ and compare to $\mathcal{D}_1$.

CONCLUDE: If $\mathcal{D}_1$ and $\mathcal{D}_1'$ are identical or have the same space of solutions, that is produce the same involutive system, then $\mathcal{D}_1$ is parametrized by $\mathcal{D}$. Otherwise, if the space of solutions of $\mathcal{D}_1'$ is smaller than that of $\mathcal{D}_1$, then $\mathcal{D}_1$ cannot be parametrized. It means that there is at least one new CC which is an autonomous element which is satisfying at least one OD or PD equation for itself, a fact showing that the initial system, considered as a control system, cannot be controllable (See [21, 35] or Zbl 179.93001 for more details and examples, in particular the study of RCL electrical circuits). We notice that we don't need to separate (η^1, η^2) into input and output.

END: In the present example, $\mathcal{D}_1'$ is defined by $d_1\eta^1 + d_2\eta^2 = \zeta'$ and we have $d_2\zeta' = \zeta$, that is $d_2\zeta' = 0$ whenever $\zeta = 0$. It follows that $\mathcal{D}$ CANNOT be parametrized. Ofc course, such a result could have been found directly in this elementary example but such a test is unavoidable in general.

In order to recapitulate, our purpose in this appendix is to explain how differential homological algebra is able to combine these two procedures.

For this we ask the reader to spend a few dollars in order to realize the next experiment and try to understand why such a fact is clearly showing that gravitational waves cannot exist as we have explained in the Introduction and in theorem 4.1 or in [36].

Example 5.1: (Double pendulum) Let us consider a thin rigid bar of length L able to slide horizontally with a position x function of time t and time derivative d . We may attach a pendulum of length l_1 , mass m_1 , (small) angle θ_1 from the vertical at one end and a pendulum of length l_2 , mass m_2 , (small) angle θ_2 from the vertical at the other end. At equilibrium, the two pendulums are fixed. If one of the pendulums is slightly moved and $l_1 \neq l_2$, moving the bar with enough skill, one can bring the full system to equilibrium, that is $x = cst, \theta_1 = \theta_2 = 0$ and we say that the full system is controllable by using the movement of the bar.

Now, if $l_1 = l_2 = l$ and one pendulum is moved while the other is untouched, then any way to stop the first wll bring the other to have the same movement and the system is not controllable that is, in particular, there is no way to stop both pendulums as before.

In order to write down the two OD equations of the system, one has to consider the Newton law applied to the inertial forces $d^2(x + l\theta)$, the gravity force mg and the tension of the thread. Projecting on the perpendicular to each pendulum to eliminate the last one, we may divide by the respective mass in order to get the two OD equations in the form of a surjective operator $\mathcal{D}_1\eta = \zeta$ with $\eta = (\eta^1 = \theta^1, \eta^2 = \theta^2, \eta^3 = x$):

$$d^2x + l_1 d^2\theta^1 + g\theta^1 = \zeta^1, \qquad d^2x + l_2 d^2\theta^2 + g\theta^2 = \zeta^2$$

Multiplying the firstOD equation by the test function λ^1 , the seond by the test fncin λ^2 , adding and integrating by parts, we obtain the adjoint operator in the form $ad(\mathcal{D}_1)\lambda = \mu$, namely:

$$\theta^2 l_2 d^2\lambda^2 + g\lambda^2 = \mu^2, x d^2\lambda^1 + d^2\lambda^2 = \mu^3$$

First of all, if $\mu^1 = \mu^2 = \mu^3 = 0$, we obtain two zero order OD equations where the second one is obtain by differentiating twice the first and substituting:

$$l_2\lambda^1 + l_1\lambda^2 = 0, \qquad (l_2/l_1)\lambda^1 + (l_1/l_2)\lambda^2 = 0$$

As the determinant of this linear system is equal to $l_1 - l_2$, it follows that $ad(\mathcal{D}_1)$ is injective if and only if $l_1 \neq l_2$.

Let us now prove that $\mathcal{D}_1$ can be parametrized if and only if such a condition is satified. First of all we notice that $\lambda^1, \lambda^2 \in j_2(\mu)$ and, using a tricky PP procedure or computer algebra, we discover that $ad(\mathcal{D}_1)$ has only one CC $ad(\mathcal{D})\mu = \nu$ of order 4. Taking the adjoint, we obtain a well defined fourth order injective parametrization in the form $\mathcal{D}\xi = \eta$ as follows:

$$\boxed{-l_1 l_2 d^4\xi - g(l_1 + l_2)d^2\xi - g^2\xi = x, \; l_2 d^4\xi + g d^2\xi = \theta^1, \; l_1 d^4\xi + g d^2\xi = \theta^2}$$

This parametrization is injective and we have the short exact adjoint sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 1 & \stackrel{D}{\longrightarrow} & 3 & \stackrel{D_1}{\longrightarrow} & 2 & \longrightarrow & 0 \\ & & & & & & & & \\ 0 & \longleftarrow & 1 & \stackrel{ad(D)}{\longleftarrow} & 3 & \stackrel{ad(D_1)}{\longleftarrow} & 2 & \longleftarrow & 0 \end{array}$$

At no moment we have ever been speaking about inputs, outputs and Kalman test !.

It remains to study the case $l_1 = l_2 = l$. Subtracting the second equation from the first while setting $\theta = \theta^1 - \theta^2$, we get $ld^2\theta = g\theta = 0$, that is the standard Od equation of a single pendulum. Of course, we notice at once that the observable θ is an autonomous element and its behaviour cannot be controlled. Equivalently, θ is generating the torsion submodule $t(M)$ of the differential module M defined by $\mathcal{D}_1$. The main fact is that now $\mathcal{D}_1$ cannot any longer be parametrized?. For this, we may multiply each OD equation by two test functions as before and integrate by parts in order to obtain:

$$ld^2\lambda^1 + g\lambda^1 = \mu^1, ld^2\lambda^2 + g\lambda^2 = \mu^2, d^2(\lambda^1 + \lambda^2) = \mu^3$$

Summing the two first OD equations while setting $\lambda = \lambda^1 + \lambda^2$, we get:

$$d^2\lambda = \mu^3,\ ld^2\lambda + g\lambda = \mu^1 + \mu^2$$

It follows that $ad(\mathcal{D}_1)$ is no longer injective with kernel $\lambda = 0 \Leftrightarrow \lambda^1 + \lambda^2 = 0$. We also obtain $g\lambda = \mu^1 + \mu^2 - l\mu^3$ and, substituting, we obtain one single CC of order 2 for $ad(\mathcal{D}_1)$, namely $ad(\mathcal{D})$ as follows:

$$d^2(\mu^1 + \mu^2 - l\mu^3) - g\mu^3 = \nu$$

Multiplying by the test function ξ and integrating by parts, we get $\mathcal{D}$:

$$\mu^1 d^2\xi = \theta^1, \mu^2 d^2\xi = \theta^2, \mu^3 - ld^2\xi - g\xi = x$$

The 3 CC are described by $\mathcal{D}'_1 \neq \mathcal{D}_1$ as follows:

$$d^2x + ld^2\theta^1 + g\theta^1 = 0, d^2x + ld^2\theta^2 + g\theta^2 = 0, \theta^1 - \theta^2 = 0$$

We recover the fact that the new CC is introducing $\theta = \theta^1 - \theta^2$ which is indeed a torsion element with autonomous OD equation $d^2\theta + g\theta = 0$.

This operator $\mathcal{D}$ of order 2 is injective and we have the short exact adjoint sequences:

$$\begin{array}{ccccccccc}
 & & & & & & 3 & & \\
 & & & & & D'_1 \nearrow & & & \\
0 & \longrightarrow & 1 & \xrightarrow{D} & 3 & \xrightarrow{D_1} & 2 & \longrightarrow & 0 \\
 & & & & & & & & \\
0 & \longleftarrow & 1 & \xleftarrow{ad(D)} & 3 & \xleftarrow{ad(D_1)} & 2 & &
\end{array}$$

It is absolutely impossible to recover all these results and understand their meaning without differential double duality and differential homological algebra.

Example 5.2: (General relativity) We now prove that the above situation is .. exactly similar to the one we have described in GR in order to explain why GW cannot exist.

When $n = 4$, let us prove that the 10 Einstein equations cannot admit a potential or, equivalently, that the Ricci operator cannot be parametrized and that the Einstein operator is therefore useless as it cannot have a mathematical origin contrary to the Ricci operator, according to the fundamental diagram II.

We shall discover that the existence of autonomous elements (torsion elements in the module framework) ... are nothing else than the $20 - 10 = 10$ components of the Weyl bundle already introduced in the fundamental diagram II, a result highly not evident at first sight !.

First of all, according to the parametrization test, to the result of the Introduction and to Theorem 4.1, we have at once the negative test procedure:

$$\begin{array}{ccccccc}
 & & & & & Riemann & 20 \\
 & & & & & \nearrow & \\
 & & 4 & \xrightarrow{Killing} & 10 & \xrightarrow{Ricci} & 10 \\
 & & & & & & \\
0 & \longleftarrow & 4 & \xleftarrow{Cauchy} & 10 & \xleftarrow{ad(Ricci)} & 10
\end{array}$$

The 5 steps are indicated as follows while taking into account that $Cauchy = ad(Killing)$, a result that can be found in any textbook of continuum mechanics or elasticity:

$Ricci \longrightarrow ad(Ricci) \longrightarrow Cauchy \longrightarrow Killing \longrightarrow Riemann$

Accordingly and in a coherent way with the previous example, the Ricci operator and thus the Einstein operator cannot be parametrized and we have $20 - 10 = 10$ generating torsion elements, that is to say elements satisfying at least one PD equation for each one. Of course, such an approach has never been followed because the fundamental diagram II is not known though it provides the splitting $Riemann = Ricci \oplus Weyl$ coming from the splitting $T^* \otimes T^* = S_2T^* \oplus \wedge^2 T^*$. My chance has been to know very well the results of A. Lichnerowicz who has been my advisor during 25 years from 1973 to his death in 1998. Indeed, I got in mind the so-called (in France !) " Lichnerowicz waves " and I am now able to give a short proof of the linearized case [18]:

Theorem 5.3: The Dalembertian of each component of the Weyl tensor is a linear differential consequence of the 10 Einstein equations $R_{ij} = 0$.

Proof. Linearizing over the Euclide or Minkowski metric ω while using the Bianchi identities and taking into account that $R_{kl,ij} = R_{ij,kl}$, we get with $d^r = \omega^{ri} d_i$:

$$d_r R_{kl,ij} + d_i R_{kl,jr} + d_j R_{kl,ri} = 0 \Rightarrow d^r R_{rl,ij} = d^r R_{ij,rl} = d_i R_{lj} - d_j R_{li}$$

$$d_{rs}R_{kl,ij} + d_{is}R_{kl,jr} + d_{js}R_{kl,ri} = 0 \Rightarrow \Box R_{kl,ij} + \omega^{rs}d_{is}R_{kl,jr} + \omega^{rs}d_{js}R_{kl,ri} = 0$$

$$\Box R_{kl,ij} = d_i(d_kR_{lj} - d_lR_{kj}) - d_j(d_kR_{li} - d_lR_{ki})$$

Finally, using the splitting formula for defining the components $\sigma^k_{l,ij} = \rho^k_{l,ij} - (\sum \rho_{rs})$ of the Weyl tensor from the components of the Riemann tensor with now $\sigma^r_{l,rj} = 0$, we obtain by linearization $\Sigma^k_{l,ij} = R^k_{l,ij} - (\sum R_{rs})$

It is absolutely impossible to extend the above result to the conformal Riemannian geometry because the analogue of the Bianchi operator is now of order 2, a result still neither known and nor even acknowledged today in the long exact sequence that we have already provided at the end of Section 3C, where $\mathcal{D}$ is the conformal Killing operator and $\mathcal{D}_1$ is the Weyl operator [15, 26, 31]. This is the main reason for which the Spencer sequences and their adjoint sequences must be used !.

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